Assignment 3 Answers

ECON 3161, Game Theory

Due: By the end of class on Thursday Oct. 11th

Directions: Answer all questions completely. Note the due date of the assignment. Late assignments will be accepted at the cost of 10 points per day, up until 11am on Tuesday Oct. 16th. At that time I will return the graded assignments and post the answers online. You may turn in assignments to me after that time so that I can check your work for you, but please realize that you will not receive a grade for the assignment. You may work in a group consisting of up to 3 members – for each group please turn in only 1 set of answers and make sure all group member names are on that set of answers. All group members will receive the same grade.

1. (20 points) Suppose that two neighbors are attempting to determine which volume level to set their speakers. Neighbor 1 has 3 choices for speaker volume: High, Medium, or Low. Neighbor 2 observes (or rather, hears) Neighbor 1’s choice of speaker volume and then gets to choose whether or not to set his speaker volume to High, Medium, or Low. The payoffs to the neighbors are as follows: If both listen to their music at a low volume, then both receive a payoff of 10. If both listen to their music at a medium volume, then both receive a payoff of 6. If both listen to their music at a high volume, then both receive a payoff of two. If one neighbor listens to his music at a medium volume while the other listens at a low volume, the payoff is 12 for the medium volume listener and 5 for the low volume listener. If one neighbor listens to his music at a high volume while the other listens at a medium volume, the payoff is 8 for the high volume listener and 1 for the medium volume listener. Finally, if one player listens to his music at a high volume and the other listens to his music at a low volume, the payoff is 15 for the high volume listener and (−1) for the low volume listener.

a (10 points) Draw the game tree (extensive form) for this sequential game.

Answer:
There are different ways in which the tree could have been drawn. I have chosen to start with the high volume and then move to low volume, but the order in which the three actions are listed at each node is not important, as long as the payoffs match up to the outcome that occurs after each player has made a decision.
b (10 points) Find a subgame perfect Nash equilibrium to this game.

Answer:
The arrows show which strategy each player would choose at each node. Starting from the end of the game:

If Neighbor 1 chose High, Neighbor 2 would choose High over Medium and Low because $2 > 1$ and $2 > -1$.

If Neighbor 1 chose Medium, Neighbor 2 would choose High over Medium and Low because $8 > 6$ and $8 > 5$.

If Neighbor 1 chose Low, Neighbor 2 would choose High over Medium and Low because $15 > 12$ and $15 > 10$.

Neighbor 1 chooses High because $2 > 1$ (which is what he would have if he chose Medium) and $2 > -1$ (which is what he would have if he chose Low).

So basically, all neighbors choose High whenever they get the chance. Note that the strategic form (or matrix) for this game would be a 27x3 matrix.

2. (20 points) Consider the following simultaneous game:

<table>
<thead>
<tr>
<th></th>
<th>Up</th>
<th>Center</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 2</td>
<td>5,6</td>
<td>8,1</td>
<td>10,4</td>
</tr>
<tr>
<td>Player 1</td>
<td>2,11</td>
<td>6,9</td>
<td>4,2</td>
</tr>
<tr>
<td></td>
<td>3,8</td>
<td>7,7</td>
<td>6,0</td>
</tr>
</tbody>
</table>

a (5 points) Find the pure strategy Nash equilibrium to this game.
Answer:
As can be seen from the best responses above, both players have a strictly dominant strategy in this game so the only PSNE is Player 1 choose Up and Player 2 choose Left.

b (5 points) Suppose that this game is repeated 17 times. Find a subgame perfect Nash equilibrium (SPNE) to this game. Is this equilibrium unique? Explain.

Answer:
The only SPNE to this game is for Player 1 to choose Up whenever he gets a chance to make a decision and Player 2 to choose Left whenever he gets a chance to make a decision. The reason is because both players have a strictly dominant strategy in the stage game AND the game is finitely repeated, so by backward induction all players will use their strictly dominant strategies in the final period and then, because the next to last period essentially becomes the last period, use their strictly dominant strategies in the next to last period, etc. Yes, the theorem from class about finitely repeated games tells us this equilibrium is unique.

c (10 points) Now suppose that this game is infinitely repeated. Suppose Player 1 uses a strategy of “Choose Middle unless a defection occurs by Player 2, where a defection is Player 2 choosing either Left or Right. If a defection is observed, choose Up forever regardless of what Player 2 chooses after the defection”. Suppose Player 2 uses a strategy of “Choose Center unless a defection occurs by Player 1, where a defection by Player 1 is a choice of Up or Down. If a defection is observed, choose Left forever regardless of what Player 1 chooses after the defection”. Find the minimum discount rate, \( \delta \), needed for each player to support these strategies as a SPNE to the infinitely repeated game.

Answer:
If P1 and P2 play the proposed strategies then P1 would receive:

\[
\sum_{i=0}^{\infty} \delta^i 6 = \frac{6}{1 - \delta}
\]

If P1 were to deviate (assuming P2 is playing the proposed strategy) then P1 would choose Up as his deviation (because \( 8 > 7 \)) and P1 would receive:

\[
8 + \sum_{i=1}^{\infty} \delta^i 5 = 8 + \frac{5\delta}{1 - \delta}
\]

In order for P1 to play the proposed strategy he would need:

\[
\frac{6}{1 - \delta} \geq 8 + \frac{5\delta}{1 - \delta}
\]

\[
6 \geq 8 (1 - \delta) + 5\delta
\]

\[
6 \geq 8 - 8\delta + 5\delta
\]

\[
3\delta \geq 2
\]

\[
\delta \geq \frac{2}{3}
\]

So P1 would need \( \delta \geq \frac{2}{3} \) in order to play the proposed strategy. Note that you could have started with: \( \Pi^1_{Coop} = 6, \Pi^1_{Deviate} = 8, \) and \( \Pi^1_{Defect} = 5 \) and used \( 6 \geq 8 (1 - \delta) + 5\delta \) to find \( \delta \geq \frac{2}{3} \). The longer way of writing it out (using the summation signs) just makes it the origin of the \( 6 \geq 8 (1 - \delta) + 5\delta \) equation clearer.

To find P2’s \( \delta \) we know that if P1 and P2 play the proposed strategies then P2 would receive:

\[
\sum_{i=0}^{\infty} \delta^i 9 = \frac{9}{1 - \delta}
\]
If P2 were to deviate (assuming P1 is playing the proposed strategy) then P2 would choose Left as his deviation (P2 certainly would not want to choose Right as the 2 he receives would be less than his cooperation payoff of 9) and would receive:

\[11 + \sum_{i=1}^{\infty} \delta^i 6 = 11 + \frac{6\delta}{1 - \delta}\]

In order for P2 to play the proposed strategy he would need:

\[\frac{9}{1 - \delta} \geq 11 + \frac{6\delta}{1 - \delta}\]

\[9 \geq 11 (1 - \delta) + 6\delta\]

\[9 \geq 11 - 11\delta + 6\delta\]

\[5\delta \geq 2\]

\[\delta \geq \frac{2}{5}\]

So P2 would need \(\delta \geq \frac{2}{5}\) in order to play the proposed strategy.

3. (20 points) Consider the following game tree:

Note that there are 3 players in this game.

a (5 points) How many subgames (including the entire game) are in this game?
There are 7 subgames in this game. Each of Player 3’s information sets starts a new subgame, as well as each of Player 2’s (the actual subgame starting from the information set with the C and D branches includes Player 3’s information sets with the GH and IJ branches, while the one starting from the information set with the E and F branches includes Player 3’s information sets with the KL and MN branches). There is also the entire game.

b (10 points) Find a subgame perfect Nash equilibrium to this game. Be sure to specify the strategies completely.

The figure above shows the marked branches (actions) each player would take at each information set. The explanation of why is below.

Player 3 would choose G over H because $9 > 8$; Player 3 would choose J over I because $3 > -5$; Player 3 would choose K over L because $9 > -6$; Player 3 would choose M over N because $2 > -6$.

Player 2 would choose C over D because $6 > 1$; Player 2 would choose E over F because $8 > 7$.

Player 1 would choose A over B because $3 > 2$.

So the SPNE to this game is:
Player 3: Uses G,J,K,M
Player 2: Uses C,E
Player 1: Uses A

c (5 points) What outcome occurs as a result of the players using the subgame perfect Nash equilibrium?

Answer:
The outcome (what we actually observe given the SPNE strategies) is A, C, G with Player 1 receiving a payoff of 3, Player 2 receiving a payoff of 6, and Player 3 receiving a payoff of 9.

4. (25 points) The Hatfields and McCoys have put aside their differences and are attempting to collude in their production of oil. If both restrict their oil supply and only produce 1000 barrels per day, each family will earn a profit of $4500. However, both families have an incentive to cheat on this agreement and produce 2000 barrels per day. If both cheat and produce 2000 barrels each family’s profits will be $3000. However, if only one firm cheats and produces 2000 barrels, then the profits to the cheating firm will be $6000, and the profits to the firm that restricts will be $2000. The matrix form of this simultaneous game is:

<table>
<thead>
<tr>
<th></th>
<th>Cheat</th>
<th>Restrict</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cheat</td>
<td>4500, 4500</td>
<td>6000, 2000</td>
</tr>
<tr>
<td>Restrict</td>
<td>2000, 6000</td>
<td>4500, 4500</td>
</tr>
</tbody>
</table>

a (5 points) Find the pure strategy Nash equilibrium (PSNE) to the one-shot game.

Answer:
Both players have a strictly dominant strategy to cheat in this game so the only PSNE is for both players to choose cheat.

b (10 points) Suppose this game is repeated infinitely. Propose a strategy such that the outcome repeated in the stage game is the (4500, 4500) outcome when both players choose restrict.

Answer:
There are a number of possible correct answers here depending on the punishment phase used by the players, but I will focus on the grim trigger strategies. Suppose the McCoys use a strategy of “Choose restrict unless a defection occurs by the Hatfields, where a defection is the Hatfields choosing cheat. If a defection is observed, choose cheat forever regardless of what the Hatfields choose after the defection”. Suppose the Hatfields use a strategy of “Choose restrict unless a defection occurs by the McCoys, where a defection by the McCoys is a choice of cheat. If a defection is observed, choose cheat forever regardless of what the McCoys choose after the defection”

c (10 points) Determine the minimum discount rate needed by each player to ensure that the set of strategies you have suggested in part b is a subgame perfect Nash equilibrium to the game.

Answer:
I am going to use the easy way here rather than typing out all the steps again. We know that $\Pi_{Cooperate}^{McCoys} = \Pi_{Cooperate}^{Hatfields} = 4500$, and that $\Pi_{Deviate}^{McCoys} = \Pi_{Deviate}^{Hatfields} = 6000$, and that $\Pi_{Defect}^{McCoys} = \Pi_{Defect}^{Hatfields} = 3000$. Since all the payoffs are the same for each player the minimum discount rate will be the same for each player and $\delta$ will need to be greater than or equal to:

$$4500 \geq 6000 - 6000\delta + 3000\delta$$

$$0 \geq 1500 - 3000\delta$$

$$3000\delta \geq 1500$$

$$\delta \geq \frac{1}{2}$$
So both players will need a discount rate of at least $\frac{1}{2}$ in order for the strategies I have used in part \textbf{b} to be an SPNE to the game.

5. (15 points) Consider the following game tree:

\begin{center}
\begin{tikzpicture}
  \node[concept] (A) {Bert} child { node[concept] (B) {Ernie} child { node[concept] (C) {Bert} child { node[concept] (D) { } } } child { node[concept] (E) { } } };
  \end{tikzpicture}
\end{center}

\textbf{a} (5 points) How many strategies does each player have?

\textbf{Answer:}

Bert has 3 information sets and two actions at each information set, so he has $2 \times 2 \times 2 = 8$ strategies (ACE, ACF, ADE, ADF, BCE, BCF, BDE, BDF). Ernie only has one information set and 2 actions at that information set so he has 2 strategies.

\textbf{b} (10 points) Find the subgame perfect Nash equilibrium (SPNE) to this game.

\textbf{Answer:}
The figure above shows the game tree with the branches (actions) marked for each player at each information set. The explanation is below.

If Bert were to choose after Ernie chooses Y he would choose D because $7 > 5$; if Bert were to choose after Ernie chooses Z he would choose E because $8 > 2$.

If Ernie were to choose between Y and Z he would choose Y because $9 > 3$.

When Bert makes his decision at the beginning of the game he chooses B because $7 > 6$.

So the SPNE to this game is:

Bert: B,D,E
Ernie: Y

While you were not asked this, the outcome would be B, Y, D with Bert receiving a payoff of 7 and Ernie receiving a payoff of 9.

**Bonus:** (10 points)

In question 2, suppose that the game is infinitely repeated. Suppose that Player 1 uses the following strategy: “Choose Middle in even periods and Down in odd periods unless a defection occurs by Player 2, where a defection is Player 2 choosing either Left or Right. If a defection is observed, choose Up forever regardless of what Player 2 chooses after the defection”. Suppose Player 2 uses a strategy of “Choose Center unless a defection occurs by Player 1, where a defection by Player 1 is a choice of Up or Down in even periods or Up or Middle in odd periods. If a defection is observed, choose Left forever regardless of what Player 1 chooses after the defection”. Find the minimum discount rate, $\delta$, needed for each player to support these strategies as a SPNE to the infinitely repeated game.

**Answer:**
If Player 1 uses this strategy (and assuming the Middle, Center outcome occurs first), Player 1’s payoff stream if both players played the proposed strategies would be:

\[ 6 + 7\delta + 6\delta^2 + 7\delta^3 + 6\delta^4 + 7\delta^5 + \ldots \]

We can break this into two pieces to make it more manageable:

\[
\begin{align*}
6 + 6\delta^2 + 6\delta^4 + 6\delta^6 + \ldots &= 6 \left(1 + \delta^2 + \delta^4 + \delta^6 + \ldots\right) \\
7\delta + 7\delta^3 + 7\delta^5 + 7\delta^7 + \ldots &= 7\delta \left(1 + \delta^2 + \delta^4 + \delta^6 + \ldots\right)
\end{align*}
\]

The key here is to realize that \(\delta^2\) is just another number that will be between 0 and 1 as long as \(\delta\) is between 0 and 1. Let’s just rewrite \(\delta^2\) as \(\beta\), so that \(\delta^2 = \beta\). Now substitute in \(\beta\) to see what happens:

\[
\begin{align*}
6 + 6\beta + 6\beta^2 + 6\beta^3 + \ldots &= 6 \left(1 + \beta + \beta^2 + \beta^3 + \ldots\right) \\
7\beta + 7\beta^3 + 7\beta^5 + 7\beta^7 + \ldots &= 7\beta \left(1 + \beta + \beta^2 + \beta^3 + \ldots\right)
\end{align*}
\]

But we know that for any number (or variable) between 0 and 1 that

\[ (1 + \beta + \beta^2 + \beta^3 + \ldots) = \frac{1}{1 - \beta} \]

Now just substitute \(\delta^2 = \beta\) back into that equation to see:

\[ (1 + \beta + \beta^2 + \beta^3 + \ldots) = \frac{1}{1 - \delta^2} \]

So we know that Player 1’s payoffs if both players use the proposed strategies will be:

\[ 6 * \frac{1}{1 - \delta^2} + 7\delta * \frac{1}{1 - \delta^2} \]

Now, what will Player 1 receive if he deviates? His one time deviation payoff will be 8 as he will choose Up (I am assuming optimal deviation here – after all, we are looking for the minimum \(\delta\)). He would then receive a payoff of 5 for each remaining period. So Player 1’s payoff if he chooses not to play the proposed strategy would be:

\[ 8 + 5\delta + 5\delta^2 + 5\delta^3 + \ldots \]
\[ 8 + \frac{5\delta}{1 - \delta} \]

Now we just compare the two and want:

\[
\begin{align*}
\frac{6}{1 - \delta^2} + \frac{7\delta}{1 - \delta^2} &\geq 8 + \frac{5\delta}{1 - \delta} \\
6 + 7\delta &\geq 8 \left(1 - \delta^2\right) + \frac{5\delta \left(1 - \delta^2\right)}{1 - \delta} \\
6 + 7\delta &\geq 8 - 8\delta^2 + \frac{5\delta \left(1 + \delta\right) \left(1 - \delta\right)}{1 - \delta} \\
6 + 7\delta &\geq 8 - 8\delta^2 + 5\delta \left(1 + \delta\right) \\
6 + 7\delta &\geq 8 - 8\delta^2 + 5\delta + 5\delta^2 \\
2\delta &\geq 2 - 3\delta^2 \\
3\delta^2 + 2\delta - 2 &\geq 0
\end{align*}
\]

Like in the bonus problem to the last assignment there isn’t much we can do here except use the quadratic formula:

\[
\frac{-2 \pm \sqrt{2^2 - 4 * 3 * (-2)}}{2 * 3} = \frac{-2 \pm \sqrt{4 + 24}}{6} = \frac{-2 \pm \sqrt{28}}{6} = \frac{-2 \pm 2\sqrt{7}}{6} = \frac{-1 \pm \sqrt{7}}{3}
\]
Now, we know that we are looking for a number between 0 and 1 so we cannot have \( \frac{1 + \sqrt{7}}{3} \) because that number is negative, so we must have \( \frac{1 + \sqrt{7}}{3} \approx 0.54858 \).

For Player 2, we know that Player 2 will receive 9 the first period and 7 the second period if both players play the proposed strategies. Using the results from above we know that Player 2’s payoff stream is:

\[
9 + 7\delta + 9\delta^2 + 7\delta^3 + 9\delta^4 + 7\delta^5 + \ldots = \frac{9}{1 - \delta^2} + \frac{7\delta}{1 - \delta^2}
\]

If Player 2 deviates in the first period he will receive 11 (he chooses Left when Player 1 chooses Middle), and then he will receive 6 forever so his payoffs would be:

\[
11 + 6\delta + 6\delta^2 + 6\delta^3 + \ldots = 11 + \frac{6\delta}{1 - \delta}
\]

Comparing these two we have:

\[
\frac{9}{1 - \delta^2} + \frac{7\delta}{1 - \delta^2} \geq 11 + \frac{6\delta}{1 - \delta}
\]

\[
9 + 7\delta \geq 11 - 11\delta^2 + 6\delta + 6\delta^2
\]

\[
5\delta^2 + \delta - 2 \geq 0
\]

Again, I don’t think we will get anywhere trying to factor that so we use the quadratic formula to find (we know it will not be the minus from \( \pm \) so I will just use + below):

\[
\frac{-1 + \sqrt{1 - 4 * 5 * (-2)}}{2 * 5} = \frac{-1 + \sqrt{41}}{10} \approx 0.54031
\]

If we assumed Player 2 deviated when it is the Down, Center outcome (so the second period when he is receiving 7 and would deviate to get 8) he would need:

\[
\frac{7}{1 - \delta^2} + \frac{9\delta}{1 - \delta^2} \geq 8 + \frac{6\delta}{1 - \delta}
\]

\[
7 + 9\delta \geq 8 - 8\delta^2 + 6\delta + 6\delta^2
\]

\[
3\delta \geq 1 - 2\delta^2
\]

\[
2\delta^2 + 3\delta - 1 \geq 0
\]

Using the quadratic formula we have that \( \delta \geq \frac{-3 + \sqrt{17}}{4} \approx 0.28078 \).

If you are just looking for the numbers, then Player 1 needs \( \delta \geq \frac{1 + \sqrt{7}}{3} \approx 0.54858 \) and Player 2 needs \( \delta \geq \frac{1 + \sqrt{41}}{10} \approx 0.54031 \).

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\(^1\)If you started with the Down, Center outcome (7, 7) payoffs, this discount rate becomes \( \frac{1 + \sqrt{7}}{3} \approx 0.434 \). Of course, a player with a discount rate of 0.45 would just deviate in the next period (go from getting the 9 payoff to the 11 payoff) because we know that the Player needs a discount rate of about 0.54858 in order to play the proposed strategy.