Assignment 4

ECON 3161, Game Theory

Due: By the end of class on Tuesday, October 30th

Directions: Answer all questions completely. Note the due date of the assignment. Late assignments will be accepted at the cost of 10 points per day, up until 11am on Thursday November 1st. At that time I will return the graded assignments and post the answers online. You may turn in assignments to me after that time so that I can check your work for you, but please realize that you will not receive a grade for the assignment. You may work in a group consisting of up to 3 members – for each group please turn in only 1 set of answers and make sure all group member names are on that set of answers. All group members will receive the same grade.

1. (50 points) Consider a simultaneous quantity choice (Cournot) game between 2 firms. Each firm chooses a quantity, $q_1$ and $q_2$ respectively. The inverse market demand function is given by $P(Q) = 1434 - 2Q$, where $Q = q_1 + q_2$. Firm 1 has total cost function $TC(q_1) = 3(q_1)^2$ and Firm 2 has total cost function $TC(q_2) = 12(q_2)^2 - 12q_2$. Each firm wishes to maximize profit.

**a** (10 points) Set up the profit function for Firm 1 and Firm 2. Remember, this is a quantity choice game.

**Answer:**
Profit functions are simply total revenue for the firm minus total cost. For each firm their respective total revenue is given by the product of their quantity and the market price. For each firm their respective total cost is given by their total cost function. So we have:

$$\Pi_1 = (1434 - 2q_1 - 2q_2)q_1 - 3q_1^2$$
$$\Pi_2 = (1434 - 2q_1 - 2q_2)q_2 - (12q_2^2 - 12q_2)$$

**b** (10 points) Find the best response functions for Firms 1 and 2.

**Answer:**
Firm 1 maximizes:

$$\Pi_1 = (1434 - 2q_1 - 2q_2)q_1 - 3q_1^2$$
$$\frac{\partial \Pi_1}{\partial q_1} = 1434 - 4q_1 - 2q_2 - 6q_1$$
$$0 = 1434 - 10q_1 - 2q_2$$
$$10q_1 = 1434 - 2q_2$$
$$q_1 = \frac{1434 - 2q_2}{10}$$
$$q_1 = \frac{717 - q_2}{5}$$

Technically, Firm 1’s best response function is $q_1 = Max \left[ 0, \frac{717 - q_2}{5} \right]$. 


Firm 2 maximizes:
\[
\Pi_2 = (1434 - 2q_1 - 2q_2)q_2 - (12q_2^2 - 12q_2)
\]
\[
\frac{\partial \Pi_2}{\partial q_2} = 1434 - 2q_1 - 4q_2 - 24q_2 + 12
\]
\[0 = 1446 - 2q_1 - 28q_2
\]
\[28q_2 = 1446 - 2q_1
\]
\[q_2 = \frac{28}{14} q_1
\]
\[q_2 = \frac{723 - q_1}{14}
\]

Technically, Firm 2’s best response function is \(q_2 = Max \left[0, \frac{723 - q_1}{14}\right]\).

c (10 points) Find the Nash equilibrium to this game.

**Answer:**

To find the Nash equilibrium to the game simply substitute one best response function into the other:

\[
q_1 = \frac{717 - q_2}{5}
\]

\[
q_1 = \frac{717 - \left(\frac{723 - q_1}{14}\right)}{5}
\]

\[
5q_1 = 717 - \left(\frac{723 - q_1}{14}\right)
\]

\[
70q_1 = 10038 - 723 + q_1
\]

\[
69q_1 = 9315
\]

\[
q_1 = 135
\]

Now to find Firm 2’s quantity simply use \(q_1 = 135\) in Firm 2’s best response function:

\[
q_2 = \frac{723 - q_1}{14}
\]

\[
q_2 = \frac{723 - 135}{14}
\]

\[
q_2 = 42
\]

So the Nash equilibrium to this game is \(q_1 = 135\) and \(q_2 = 42\).

d (10 points) Find the (1) total market quantity, (2) price, and (3) profit for each firm.

**Answer:**

The market quantity is \(Q = q_1 + q_2\), so \(Q = 177\). The market price is \(P (Q) = 1434 - 2Q\), so \(P = 1080\). Each firm’s profit is:

\[
\Pi_1 = P * q_1 - TC (q_1)
\]

\[
\Pi_1 = 1080 * 135 - 3 * 135^2
\]

\[
\Pi_1 = 91125
\]

and

\[
\Pi_2 = P * q_2 - TC (q_2)
\]

\[
\Pi_2 = 1080 * 42 - (12 * 42^2 - 12 * 42)
\]

\[
\Pi_2 = 24696
\]
e (10 points) Assume Firm 1 is the only producer in the market now. Find the Firm 1’s monopoly (1) quantity, (2) price, and (3) profit.

Answer:
If Firm 1 is the only producer then Firm 1 is a monopolist so that $q_2 = 0$. There are many ways to go about this, the most direct being to set up Firm 1’s profit function (assuming $q_2 = 0$) and solve for the monopoly quantity:

$$\Pi_1 = (1434 - 2q_1)q_1 - 3q_1^2$$
$$\frac{d\Pi_1}{dq_1} = 1434 - 4q_1 - 6q_1$$
$$0 = 1434 - 10q_1$$
$$10q_1 = 1434$$
$$q_1 = 143.4$$

We now know that Firm 1’s monopoly quantity is 143.4 (we could also have used Firm 1’s best response function from part b to figure this out), with price being equal to

$$P(Q) = 1434 - 2Q$$
$$P(143.4) = 1434 - 2 \times 143.4$$
$$P = 1147.2$$

Now knowing both $P$ and $q_1$ we can find Firm 1’s profit:

$$\Pi_1 = P \times q_1 - TC(q_1)$$
$$\Pi_1 = 1147.2 \times 143.4 - 3 \times (143.4)^2$$
$$\Pi_1 = 164508.48 - 61690.68$$
$$\Pi_1 = 102817.8$$

2. (25 points) The citizens of Circleburg live in a city that is laid out in a perfect circle. The circumference of the circle is 12 miles. Residents live in houses which are evenly spaced (or uniformly distributed) over the 12 miles. There are three competing gas stations, Chi Station, Delta Station, and Tau Station. They are attempting to determine where to locate their respective stations. They know that residents of Circleburg will go to the gas station closest to their home. Assume that gas stations are concerned with maximizing the number of customers who visit their station. You may want to use a diagram to aid you when answering the questions. A Nash Equilibrium for this game is a set of locations for the gas stations. Note that gas stations may locate at the same point on the circle.

Note: Customers and gas stations can only locate on the perimeter of the circle. Assume that the interior of the circle is a huge chasm or a steep mountain. Someone always tries to locate stations/customers in the middle of the circle - do NOT do that. Hint: It may help to think of the circle as the face of a clock.

a (5 points) Find a pure strategy Nash equilibrium to this game where all firms receive the same number of customers. Explain why this is a PSNE.

Answer:
Any set of locations such that the firms are equally spaced by 4 units is a PSNE in which the number of customers is the same for all firms. An example would be the 12 o’clock position for Chi, the 4 o’clock position for Delta, and the 8 o’clock position for Tau. Chi gets all the customers from 10-2, Delta gets all the customers from 2-6, and Tau gets all the customers from 6-10, so each gets 1/3 of the market. Note that this is a PSNE because if any one firm decides to move it will either still get 1/3 of the customers (if it moves on the bigger part of the perimeter of the circle between the other two firms – if Delta moves from 4 to 5 with Chi still at 12 and Tau still at 8) or it will get less (if it switches "sides" to move between the smaller distance of the two firms – for instance if Delta moves from 4 to 10 with Chi at 12 and Tau at 8 then Delta receives only 1/6 of the customers).
b (10 points) Find a pure strategy Nash equilibrium to this game where all firms do NOT receive the same number of customers. Explain why this is a PSNE.

Answer:
There are many PSNE in this game as well. Consider a starting set of locations such as Chi at 12 o'clock, Delta at 5 o'clock, and Tau at 7 o'clock. Under this starting set of locations Chi receives all customers from 9.5 to 12 (as 9.5 is the halfway point between 12 and 7) and from 12 to 2.5 (as 2.5 is the halfway point between 12 and 5). Delta receives all customers from 2.5 to 6. Tau receives all customers from 6 to 9.5. So Chi has \( \frac{5}{12} \) (or \( \frac{10}{24} \)) of the customers while Delta and Tau each have \( \frac{3.5}{12} \) (or \( \frac{7}{24} \)).

We are going to use what we know from part a here. We know that any firm that moves between the other two firms on the part of the circle it is already on will not change its own number of customers. With Chi at 12 and Delta at 5, we know that if Tau moves anywhere between 5 and 12 (so 6, 7, 8, 9, 10, 11 or any spot in between) the Tau will still receive \( \frac{7}{24} \) of the customers. If Tau switches "sides" on the circle and is now located somewhere between 12 and 5 (so 1, 2, 3, 4 or any of the spots between those numbers) then Tau will receive LESS customers (if Tau moved to 2 it would receive all the customers from 1 to 3.5, so it would receive \( \frac{2.5}{12} = \frac{5}{24} \) of the customers, which is less than \( \frac{7}{24} \)). The same logic applies to both Chi and Delta as well, and no firm can deviate from the initial set of locations.

While the proof is a little complicated, the basic idea is fairly simple. If you have 3 firms any set of locations will be a PSNE provided that, holding the locations of two firms constant, the third firm is located on the arc of the circle that has a larger distance between the two firms who have locations being held constant. Then the third firm cannot "jump to the other arc" and grab more customers. So something like Chi at 12, Delta at 1, and Tau at 2 would NOT be an equilibrium. The length of the arc between Chi and Tau on which Delta is located is only \( \frac{1}{2} \) of the perimeter of the circle (and Delta receives only \( \frac{1}{12} \) of the customers); the length of the arc between Chi and Tau on which Delta is NOT located is \( \frac{5}{6} \). So that set of locations cannot be an equilibrium because Delta could switch to the \( \frac{5}{6} \) side and receive a strictly higher number of customers (if Delta moved to 7 it would now receive \( \frac{2.5}{12} \) of the customers). Technically, all the PSNE in part a are just special cases of this general result.

c (10 points) The citizens of Circleburg have decided to outlaw backward thinking, which includes counterclockwise driving. Now, residents of Circleburg will stop at the first gas station they see when they leave their home. If there is a PSNE to this game, find it. If there are multiple PSNE, describe the set of equilibria. If there are no PSNE, explain why there are none.

Answer:
There are no PSNE to this game. Essentially, each station wants to be "just counterclockwise" of the other stations. Consider the following 3 cases:

Case 1: All stations locate at the same point (say 12 o'clock). Any station has the incentive to locate at just before the 12 o'clock position to capture most of the market.

Case 2: Two stations locate at one point and a third locates at a different point. Again, either of the two stations at the same point has the incentive to move a little bit counterclockwise to take the entire part of the market it is currently sharing with the other firm. Also, the station at the different location has the incentive to move just counterclockwise of the two stations at the same location to take the whole market.

Case 3: All stations locate at different points. The best thing any one station can do is to locate a little bit counterclockwise of the station that has the biggest mass of consumers. If there is one station at 12 o'clock and one at 5 o'clock, then the 3rd station wants to locate just a little before 12 o'clock. But if that is the case then either of the other firms would like to locate just counterclockwise of the 3rd firm.

Basically what happens is all the firms wish to play a game of leapfrog counterclockwise around the circle.
3. (25 points) Consider two firms engaged in price (Bertrand) competition, such that the firm that charges the lowest price produces the entire market quantity. If the 2 firms charge the same price they split the market quantity evenly. This means that each firm has the following demand function:

\[
\begin{array}{c|c|c}
\text{if } p_1 > p_2 & q_1 & q_2 \\
\hline
0 & \frac{500-p_1}{2} & \frac{500-p_2}{2} \\
\hline
\frac{500-p_1}{2} & \frac{500-p_1}{2} & 0 \\
\end{array}
\]

Each firm has total cost equal to \(TC(q_i) = 20q_i\), so that each firm has a constant marginal and average cost of production of 20.

a (10 points) Suppose that the pricing choices are made simultaneously. Find the pure strategy Nash equilibrium to this game.

Answer:
This is the same Bertrand pricing game from class, only now each firm’s marginal cost is equal to 20, so the PSNE to this game is \(p_1 = p_2 = 20\). Note that both firms receive a profit of zero if these strategies are used. If either firm lowers its price it will capture the entire market but it will be charging a price less than its cost so it will be making a loss. So neither firm will lower its price. If either firm raises its price then the other firm captures the entire market and the firm that raised its price would still receive a profit of zero. So neither firm has an incentive to raise its price. Because no firm can unilaterally deviate from the set of strategies, we have that \(p_1 = p_2 = 20\) is a PSNE to the game.

Now suppose that Firm 1 announces the following policy: We are going to charge $260 for our product. If any customer finds a lower price for this product than $260 then tell us and we will only match that price but offer a refund equal to the difference in the two prices. For instance, if another firm charges $240, we will only charge $220 (take $260 - $240 = $20 and then deduct another $20 so that the total amount deducted from our price of $260 is $40). This is known as a price-beating policy.

b (10 points) Find Firm 2’s best response to Firm 1’s policy announcement. Reminder: This is only a one period game.

Answer:
What happens if Firm 2 chooses a price below $260? Then it will receive a profit of zero, because, according to Firm 1’s policy, Firm 1 will end up charging a price below Firm 2 and Firm 2 will receive no sales. What happens if Firm 2 prices above $260? Then Firm 1 receives all the sales. But if Firm 2 prices just at $260 then Firm 1 will end up splitting the market with Firm 1 and both will make a positive profit. So Firm 2’s best response is to choose \(p_2 = 260\).

c (5 points) Given the best response you found for Firm 2 in part b, is that best response and Firm 1’s strategy (announced policy and price choice of $260) a pure strategy Nash equilibrium to the game? Explain.

Answer:
We know that Firm 2 is best responding to Firm 1’s strategy, but technically Firm 1 could choose a different price than $260 (say $259) and be better off. This would then lead to Firm 2 charging the same price as Firm 1, Firm 1 undercutting again, and so on and so forth until both end up pricing at marginal cost again as in part a.

Bonus:
(5 points) In game 3, suppose that Firm 1 announced a policy where they merely met Firm 2’s price (a price-matching policy). Does this affect the equilibrium to the game when compared to the price-beating policy? Explain why or why not.

Answer:
It does not—it really takes BOTH firms to implement price-matching (or price-beating) policies in order to maintain a high price because if only one firm implements the price-matching (or price-beating) policy, then that firm has an incentive to undercut. But if both firms implement a price-matching policy then neither firm has the incentive to undercut the other (because the other firm’s price just drops with the firm that lowered price) and so a collusive outcome can be maintained. Think about that for a minute—firms can maintain high prices by using a strategy that looks like it will be better for the consumer. As I mentioned when we discussed voting mechanisms, it is always good to consider the incentives of the players when considering which policies to implement.