Directions: Answer each question completely. If you cannot determine the answer, explaining how you would arrive at the answer may earn you some points.

1. (30 points) Consider a simultaneous game between two firms where each can choose one of three different quantity levels: 30, 50, and 70. Their payoffs are in the following matrix:

<table>
<thead>
<tr>
<th>Firm 2</th>
<th>q2 = 30</th>
<th>q2 = 50</th>
<th>q2 = 70</th>
</tr>
</thead>
<tbody>
<tr>
<td>q1 = 30</td>
<td>1800,1800</td>
<td>1500,2000</td>
<td>900,1800</td>
</tr>
<tr>
<td>q1 = 50</td>
<td>2000,1500</td>
<td>1600,1600</td>
<td>800,1200</td>
</tr>
<tr>
<td>q1 = 70</td>
<td>1800,900</td>
<td>1200,800</td>
<td>0,0</td>
</tr>
</tbody>
</table>

a (5 points) Define the term strictly dominant strategy. Does either firm have a strictly dominant strategy? If so which firm and which strategy?

Answer:
A strictly dominant strategy is a strategy that always provides a strictly greater payoff than any other strategy the player could choose given the strategy the other player is choosing. In this game, 2000 is both Firm 1’s and Firm 2’s highest payoff, so if any strategy is going to be strictly dominant for Firm 1 it has to be q1 = 50 and for Firm 2 it would have to be q2 = 50. We can see that neither strategy is strictly dominant because while q1 = 50 and q2 = 50 is best when the other firm chooses either 30 or 50, when the other firm chooses q1 = 70 then choosing qj = 30 is best.

b (5 points) Looking at the entire 3x3 game (and just the entire game), does either firm have any strictly dominated strategies? If so, which firm and which strategy (or strategies) are strictly dominated and which strategy (or strategies) are they strictly dominated by?

Answer:
Yes, both firms have a strictly dominated strategy. For Firm 1, q1 = 70 is strictly dominated by q1 = 50 (2000>1800, 1600>1200, 800>0). For Firm 2, q2 = 70 is strictly dominated by q2 = 50 (2000>1800, 1600>1200, 800>0).

c (10 points) Use the iterated elimination of dominated strategies (IEDS) to reduce the matrix as far as possible. Explain the steps you use to reduce this matrix.

Answer:
From part b we know that we can eliminate q1 = 70 because it is strictly dominated by q1 = 50 as well as q2 = 70 because it is strictly dominated by q2 = 50. This leaves a 2x2 matrix:

<table>
<thead>
<tr>
<th>Firm 2</th>
<th>q2 = 30</th>
<th>q2 = 50</th>
</tr>
</thead>
<tbody>
<tr>
<td>q1 = 30</td>
<td>1800,1800</td>
<td>1500,2000</td>
</tr>
<tr>
<td>q1 = 50</td>
<td>2000,1500</td>
<td>1600,1600</td>
</tr>
</tbody>
</table>

With this 2x2 matrix we can see that now q1 = 30 is strictly dominated by q1 = 50 and also q2 = 30 is strictly dominated by q2 = 50. Thus we can remove q1 = 30 and q2 = 30 to yield a 1x1 "matrix":

1
Firm 2
$q_2 = 50$
Firm 1
$q_1 = 50
\frac{1600, 1600}{1600, 1600}$
This is the furthest the matrix can be reduced.

d (5 points) Find all pure strategy Nash equilibria (PSNE) to this game.

**Answer:**

Since the matrix can be reduced to a single outcome cell there is only one PSNE to this game: Firm 1 choose $q_1 = 50$ and Firm 2 choose $q_2 = 50$.

e (5 points) Will there be a mixed strategy Nash equilibrium to this game? If so find it; if not, explain why not.

**Answer:**

There is no MSNE in this game. By using IEDS we can reduce the matrix to a single outcome cell so if we attempt to find an MSNE to the game (either the 3x3 or 2x2 after $q_1 = 70$ and $q_2 = 70$ have been eliminated) we will be attempting to make players indifferent over strategies that are strictly dominated, which cannot happen.

2. (25 points) Consider the following game:

\[
\begin{array}{ccc}
& Y & Z \\
P1 & W & 2, 2 \\
& X & 5, 5 & 3, 3 \\
\end{array}
\]

a (10 points) Find all pure strategy Nash equilibria (PSNE) to this game.

**Answer:**

The PSNE can be determined by using the method of best responses as in the game above. There are two PSNE: (1) P1 choose X, P2 choose Y and (2) P1 choose W and P2 choose Z.

b (10 points) Find the mixed strategy Nash equilibrium (MSNE) to this game.

**Answer:**

The MSNE can be found be setting the expected value of each player’s pure strategies equal to each other and then finding the probabilities of the other player that cause the expected value to be equal. For P1’s probabilities, letting $w$ be the probability he chooses strategy W, we need:

\[
E_2 [Y] = E_2 [Z]
\]

\[
2w + 5(1 - w) = 3w + 2(1 - w)
\]

\[
3 - 3w = w
\]

\[
3 = 4w
\]

\[
\frac{3}{4} = w
\]

So P1 would choose W with probability $\frac{3}{4}$ and X with probability $\frac{1}{4}$. Note that P2’s expected value of choosing strategy Y or Z when P1 uses these probabilities is $\frac{11}{4}$.

For P2’s probabilities, letting $y$ be the probability he chooses strategy Y, we need:

\[
E_1 [W] = E_1 [X]
\]

\[
2y + 3(1 - y) = 5y + 1(1 - y)
\]

\[
2 - 2y = 3y
\]

\[
\frac{2}{5} = y
\]
So P2 would choose Y with probability $\frac{2}{5}$ and Z with probability $\frac{3}{5}$. Note that P1’s expected value of choosing strategy W or X when P2 uses these probabilities is $\frac{13}{5}$.

So the MSNE is P1 choose W with probability $\frac{3}{4}$ and X with probability $\frac{1}{4}$ and P2 choose Y with probability $\frac{2}{5}$ and Z with probability $\frac{3}{5}$.

c (5 points) Can any of the equilibria (either pure or mixed) be eliminated using either the equilibrium payoff dominance criterion or the undominated Nash equilibrium criterion? If so, which ones and by which criteria?

Answer:

None of the equilibria involve using a weakly dominated strategy so no equilibria can be removed by the undominated Nash equilibrium criterion. The table below lists the payoffs for each player to each equilibrium:

<table>
<thead>
<tr>
<th>Eq.</th>
<th>P1 payoff</th>
<th>P2 payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X, Y$</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$W, Z$</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>MSNE</td>
<td>$\frac{13}{5}$</td>
<td>$\frac{11}{4}$</td>
</tr>
</tbody>
</table>

Since both players are strictly better off under the PSNE of X, Y, we can eliminate the other PSNE ($W, Z$) and the MSNE by equilibrium payoff dominance.

3. (25 points) Consider a simultaneous game between two firms where each can choose one of three different price levels: $10, 50, \text{ and } 70$. Their payoffs are in the following matrix:

<table>
<thead>
<tr>
<th>Firm 2</th>
<th>$p_2 = 10$</th>
<th>$p_2 = 50$</th>
<th>$p_2 = 70$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1 = 10$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$p_1 = 50$</td>
<td>0</td>
<td>1600</td>
<td>3200</td>
</tr>
<tr>
<td>$p_1 = 70$</td>
<td>0</td>
<td>3200</td>
<td>1800, 1800</td>
</tr>
</tbody>
</table>

a (10 points) Does any firm have any weakly dominated strategies? If so, which firm and which strategy (or strategies) are weakly dominated and which strategy (or strategies) are they weakly dominated by?

Answer:

Both firms have weakly dominated strategies in this game. For Firm 1, $p_1 = 10$ is weakly dominated by both $p_1 = 50$ and $p_1 = 70$. Also, $p_1 = 70$ is weakly dominated by $p_1 = 50$. The same is true for Firm 2: $p_2 = 10$ is weakly dominated by both $p_2 = 50$ and $p_2 = 70$, and $p_2 = 70$ is weakly dominated by $p_2 = 50$.

b (10 points) Find all pure strategy Nash equilibria (PSNE) to this game.

Answer:

The PSNE can be found by using the method of finding the best responses, as shown in the matrix above. There are two PSNE: (1) Firm 1 choosing $p_1 = 10$ and Firm 2 choosing $p_2 = 10$ and (2) Firm 1 choosing $p_1 = 50$ and Firm 2 choosing $p_2 = 50$.

c (5 points) Considering only the PSNE, can any equilibrium be eliminated using either the undominated Nash equilibrium or equilibrium payoff dominance criteria? If so, which equilibrium and by which criterion? Explain why.

Answer:

The PSNE where both firms choose a price of $10$ can be eliminated by either of the criteria. Note that choosing $10$ is weakly dominated by choosing $50$, so by undominated Nash equilibrium firms should choose the equilibrium where both choose a price of $50$. Also, both firms receive a strictly higher payoff in the PSNE when both choose $50$, so by equilibrium payoff dominance we can eliminate the equilibrium where both choose $10$. 

3
4. (20 points) Consider the following 3 person game:

<table>
<thead>
<tr>
<th></th>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>Player 1</td>
<td>A</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>1</td>
</tr>
</tbody>
</table>

Player 3

<table>
<thead>
<tr>
<th></th>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>Player 1</td>
<td>A</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>1</td>
</tr>
</tbody>
</table>

a (5 points) Does any player have a strictly or weakly dominant strategy? If so which player and which strategy?

Answer:
The best responses are marked in the matrix above. Note that every payoff for strategy B for Player 2 (in both matrices) is marked. Strategy B for Player 2 does strictly better than strategy A for Player 2 when: (1) Player 1 chooses B and Player 3 chooses A, (2) Player 1 chooses A and Player 3 chooses B, and (3) both Players 1 and 3 choose B. When both Players 1 and 3 choose A then strategy A and B (for Player 2) give the same payoff of 1. So B is a weakly dominant strategy for Player 2. There are no other weakly dominant strategies.

b (10 points) Find all pure strategy Nash equilibria (PSNE) to this game.

Answer:
There are two PSNE to this game: (1) All 3 players choose B and (2) all 3 players choose A.

c (5 points) Considering only the PSNE, can any equilibrium be eliminated using either the undominated Nash equilibrium or equilibrium payoff dominance criteria? If so, which equilibrium and by which criterion? Explain why.

Answer:
The equilibrium when all 3 players choose B can be eliminated by equilibrium payoff dominance because Players 1 and 3 are strictly better off when all choose A and Player 2 is no worse off when all choose A than when all choose B.

Also note that the equilibrium when all 3 players choose A can be eliminated by undominated Nash equilibrium because Player 2 is using a weakly dominated strategy in this equilibrium.

Bonus:
(5 points) Consider the game in question 3. Will there be any mixed strategy Nash equilibria in this game? Explain why or why not and find the MSNE if you believe one exists.

Answer:
I’m going to repost the matrix so it is here:

<table>
<thead>
<tr>
<th></th>
<th>Firm 1</th>
<th>Firm 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>p1 = $10</td>
<td>p2 = $10</td>
</tr>
<tr>
<td>p1 = $10</td>
<td>0, 0 , 1600</td>
<td>0, 0</td>
</tr>
<tr>
<td>p1 = $50</td>
<td>0, 0</td>
<td>1600</td>
</tr>
<tr>
<td>p1 = $70</td>
<td>0, 0</td>
<td>0, 3200</td>
</tr>
</tbody>
</table>

This is an odd game where there is a weakly dominant strategy for both firms where they each choose $50 and there are no strictly dominated strategies for any firm. Suppose we try to find an MSNE over all 3 outcomes. Let $a$ be the probability that Firm 1 chooses $10$, $b$ be the probability it chooses $50$, and $c$ be the probability it chooses $70$. What I am going to do (you’ll see why shortly) is set this up letting $a = 1 - b - c$. If I use the following two equations:

\[ E_2[10] = E_2[50] \]

and

\[ E_2[10] = E_2[70] \]
I get:

\[
\begin{align*}
E_2[\$10] &= E_2[\$50] \\
0 \cdot (1 - b - c) + 0 \cdot b + 0 \cdot c &= 0 \cdot (1 - b - c) + 1600 \cdot b + 3200 \cdot c \\
0 &= 1600b + 3200c \\
-3200c &= 1600b \\
-2c &= b
\end{align*}
\]

(I used \( a = 1 - b - c \) because I knew \( a \) would be multiplied by 0). Now there’s a problem already, that \(-2c = b\) and the only way this can happen is if \( b = c = 0 \). Let’s look at the second equation:

\[
\begin{align*}
E_2[\$10] &= E_2[\$70] \\
0 \cdot (1 - b - c) + 0 \cdot b + 0 \cdot c &= 0 \cdot (1 - b - c) + 0 \cdot b + 1800 \cdot c \\
0 &= 1800c \\
0 &= c
\end{align*}
\]

We have now confirmed that \( c = 0 \), which means that \( b = 0 \), which means that \( a = 1 \). But this leads us right back to our PSNE where both players choose $10.

What if we wanted to try to make a mixture over only two strategies (say $10 and $50, or $10 and $70, or $50 and $70)? If we try to mix over only $50 or $70 (and leave out the $10 strategy) then playing $70 is strictly dominated by $50 and we can’t make a mixture that adheres to the laws of probability. If we try to mix over $10 and $50 (which makes the most sense since this involves the two strategies that determine the PSNE), then we get:

\[
\begin{align*}
E_2[\$10] &= E_2[\$50] \\
0 &= 0 \cdot (1 - b) + 1600 \cdot b \\
0 &= b
\end{align*}
\]

So we are right back to where we started with both firms choose a price of $10.

If we try to mix over $10 and $70 (which may make the least sense since we are omitting the weakly dominant strategy), we still end up back at the PSNE where both players choose $10.

\[
\begin{align*}
E_2[\$10] &= E_2[\$70] \\
0 &= 0 \cdot (1 - c) + 1600 \cdot c \\
0 &= c
\end{align*}
\]

Now we have exhausted all the possibilities and we still cannot find an MSNE that is not a PSNE. This is because there is a PSNE involving both firms using weakly dominated strategies. In class I mentioned that there are typically an odd number of equilibria in games - however, this is one of the special cases where we actually have an even number of equilibria as there are no MSNE which are not PSNE.