Problem Set 1 Answers

BPHD8110-001

February 2, 2017

Problem set 1 - in general these are meant to be easy questions to develop intuition

1. Show that any strictly dominant strategy in game \([I, \{\Delta(S_i), \{u_i(\cdot)\}\}]\) must be a pure strategy.

Answer:

Proof by contradiction. Assume that the strictly dominant strategy is a nondegenerate mixed strategy, \(\sigma_i\), over \(N\) pure strategies. Then we must have:

\[
 u_i(\sigma_i, s_{-i}) > u_i(s_i^*, s_{-i}) \quad \forall \ s_i^* \in S_i \text{ and } \forall \ s_{-i} \in S_{-i}.
\]

In particular,

\[
 u_i(\sigma_i, s_{-i}) > u_i(s_j^i, s_{-i}) \quad \forall \ j = 1, ..., N.
\]

All this means is that the utility of playing \(\sigma_i\) is greater than the utility of playing any pure strategy used in \(\sigma_i\) against some other players’ strategies \(s_{-i}\). This implies that:

\[
 u_i(\sigma_i, s_{-i}) > \sum_{j=1}^{N} \left[ \sigma_i(s_j^i) \ast u_i(s_j^i, s_{-i}) \right] = u_i(\sigma_i, s_{-i}).
\]

This is clearly a contradiction. It’s not a particularly explanatory contradiction, but it is a contradiction. Intuitively, if one is playing a mixed strategy, then this suggests that the pure strategies over which one mixes must each do better against some set of the other players’ strategies.

2. There are 10 people each with 5 tokens. The players will play a simultaneous game. Players can either keep the tokens for themselves or contribute tokens to the pot. Any tokens kept by the player pay the player who kept the tokens, and only that player, $1 each. So if a player keeps 4 tokens he receives $4, but other players receive nothing unless they also kept tokens. The tokens in the pot are counted and every player in the group receives $0.5 for each token in the pot. So, if there are 12 tokens in the pot then each player receives a payout of $6 from the pot, regardless of how many he or she contributed. Thus the player’s total payoff is $1 times every token kept plus $0.5 times every token in the pot.

a Do any of the players have a strictly or weakly dominant strategy?

Answer:

All players have a strictly dominant strategy to keep all of their tokens. To see this, note that a token kept by the player is worth $1 to that player, while a token contributed is worth $0.5. While the best AGGREGATE outcome is for all players to contribute all their tokens, if 9 players contributed all 5 tokens then the best response for the 10th player is to keep all 5 tokens. If 9 players contribute, then the 10th player receives $22.50 from the pot. If the 10th player contributes all 5 of his tokens, then the 10th player receives $25 from the pot and $0 from the kept tokens so the 10th player (as well as all other players) receive $25 total. But if the 10th player keeps all 5 tokens, then the 10th player receives $22.50 from the pot and $5 from the kept tokens, for a total of $27.50. Since $27.50>$25, the best response is to keep all tokens. This is true for any contribution level by the other players.
b Find a Nash equilibrium to this game.

**Answer:**
Because all players have a strictly dominant strategy, the only NE to this game will be for all 10 players to keep their 5 tokens.

*Now, consider the case of those 10 people, each with their 5 tokens, attempting to privately finance a library.* The pot can now be thought of as the library fund. If the total contribution to the pot is greater than or equal to $20 (so there are at least 20 tokens in the pot), then the library is funded. If there are less than 20 tokens then the library is not funded. If the library is funded then each player receives a benefit of $15. In addition to that $15, any remaining tokens that they have also add $1 to their benefit. So if the library is funded and Player 1 kept 5 tokens, Player 1 receives a benefit of $20. If the library is not funded then players only receive a benefit equal to the amount of tokens that they kept.

c Find a PSNE where the library is NOT funded.

**Answer:**
Because no player can fund the library singlehandedly, if no one else contributes to the fund, then the best response is for a particular player to not contribute to the fund. Thus, this NE is similar to the one in part b, where all players keep all 5 of their tokens.

d There are many PSNE where the library is funded. Describe the set of PSNE where the library will be funded.

**Answer:**
All the PSNE involve EXACTLY 20 tokens being contributed to the library fund. These tokens may be contributed in any number of ways. The least equitable would be for 4 players to contribute all 5 tokens. The most equitable would be for each player to contribute 2 tokens. Regardless, exactly 20 should be contributed. The library will still be funded if more than 20 are contributed, but because any tokens in excess of 20 do not help the library become funded, contributing those extra tokens only reduces a player’s payoff. Thus, if 21 tokens were contributed, some player would have an incentive to take 1 token back (but only 1).

3. Show that the two-player game below has a unique equilibrium:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>C</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>U</strong></td>
<td>1</td>
<td>-2</td>
<td>-2</td>
</tr>
<tr>
<td><strong>M</strong></td>
<td>-2</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td><strong>D</strong></td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

**Player 2**

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>C</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>U</strong></td>
<td>1</td>
<td>-2</td>
<td>-2</td>
</tr>
<tr>
<td><strong>M</strong></td>
<td>-2</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td><strong>D</strong></td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

**Player 1**

**Answer:**
The best responses are marked in the game above. Note that the only PSNE is P1 D and P2 R. Now that we know what the PSNE is, and we are told that the game has a unique equilibrium, we need to rule out MSNE.

Let’s suppose that Player 2 wanted to mix over L, C, and R with probabilities $p_L$, $p_M$, and $p_R$. Player 1 would then receive:

\[
E_1[U] = p_L - 2p_M \\
E_1[M] = -2p_L + p_M \\
E_1[D] = p_R
\]
Now we have that:

\[ E_1[U] = E_1[M] \]
\[ p_L - 2p_M = -2p_L + p_M \]
\[ 3p_L = 3p_M \]
\[ p_L = p_M \]

Now, before continuing, consider the payoff to each of P1’s strategies under this result (whether \( p_L = p_M = 0.5 \) or \( p_L = p_M = 0.3 \) or any other acceptable probability). The expected value to choosing \( U \) and \( M \) will be negative, and the expected value to choosing \( D \) will be (at worst) zero. Thus, P1 would choose \( D \) regardless of the mixed strategy chosen by P2, which would then lead to P2 choosing \( R \) (because \( D, R \) is the PSNE). We can rule out P1 mixing over all 3 strategies by similar logic.

Note that the condition \( p_L = p_M \) will need to hold regardless of whether or not strategy \( R \) is used in P2’s mixed strategy. Thus, P2 cannot mix over just \( L \) and \( C \). By similar logic, P1 cannot mix over just \( U \) and \( M \).

Suppose P2 chose to mix over \( L \) and \( R \). Now, P1 has no reason to use \( M \) as part of a mixed strategy because it is strictly dominated by both \( D \). When \( M \) is removed, note that strategy \( L \) is strictly dominated by \( R \), so P2 will not mix over \( L \) and \( R \). Similar logic rules out P2 mixing over \( C \) and \( R \), and also P1 mixing over \( U \) and \( D \) or \( M \) and \( D \).

Now that we have ruled out all potential MSNE, we have proved that there is a unique NE to the game. We used "proof by exhaustion" to show this result,\(^1\) so the proof is not very elegant, but it works in this case.

4. This question will test your understanding of Proposition 8.D.1 in MWG. Find all pure and mixed strategy Nash equilibria to the following game. If there are none of either type explain why there are none:

<table>
<thead>
<tr>
<th></th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
<th>J</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A</strong></td>
<td>18</td>
<td>11</td>
<td>9</td>
<td>12</td>
<td>6.1</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>6.6</td>
<td>7.8</td>
<td>5.7</td>
<td>9</td>
<td>11</td>
</tr>
<tr>
<td><strong>C</strong></td>
<td>9.0</td>
<td>4.5</td>
<td>14.4</td>
<td>4.10</td>
<td>5.16</td>
</tr>
<tr>
<td><strong>D</strong></td>
<td>3.4</td>
<td>3.6</td>
<td>2.3</td>
<td>6.7</td>
<td>1.9</td>
</tr>
<tr>
<td><strong>E</strong></td>
<td>0.0</td>
<td>4.2</td>
<td>7.1</td>
<td>7.4</td>
<td>8</td>
</tr>
</tbody>
</table>

**Answer:**

The best responses are marked in the matrix above. This leads to three PSNE:

1. P1 choose A, P2 choose G; 2. P1 choose B, P2 choose I; 3. P1 choose E, P2 choose J.

What you should notice is that strategies F and H are both strictly dominated by G, and strategy D is strictly dominated by A, so all three of those can be removed leaving:

<table>
<thead>
<tr>
<th></th>
<th>G</th>
<th>I</th>
<th>J</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A</strong></td>
<td>12</td>
<td>8.0</td>
<td>7.1</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>7.8</td>
<td>11</td>
<td>4.5</td>
</tr>
<tr>
<td><strong>C</strong></td>
<td>4.5</td>
<td>4.10</td>
<td>5.16</td>
</tr>
<tr>
<td><strong>E</strong></td>
<td>4.2</td>
<td>7.4</td>
<td>8</td>
</tr>
</tbody>
</table>

Now we can see that strategy C, once H has been removed, is strictly dominated by A so it too can be removed leaving:

\(^1\)Proof by exhaustion means that we examined all of the potential equilibria and were able to show that they could not be an equilibrium. We "exhausted" all of the possibilities.
Player 2

<table>
<thead>
<tr>
<th></th>
<th>G</th>
<th>I</th>
<th>J</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>9</td>
<td>12</td>
<td>8</td>
</tr>
<tr>
<td>B</td>
<td>7.8</td>
<td>9</td>
<td>11</td>
</tr>
<tr>
<td>E</td>
<td>4.2</td>
<td>7.4</td>
<td>8.6</td>
</tr>
</tbody>
</table>

Player 1

<table>
<thead>
<tr>
<th></th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>8.0</td>
</tr>
<tr>
<td>E</td>
<td>7.1</td>
</tr>
</tbody>
</table>

Now let’s try to find an MSNE. Let \( g, i, \) and \( j = 1 - g - i \) be the probabilities that Player 2 uses to choose strategies G, I, and J respectively. We need:

\[
\]

Focusing on:

\[
E[A] = E[B]
\]
\[
9g + 8i + 7(1 - g - i) = 7g + 9i + 4(1 - g - i)
\]
\[
9g + 8i + 7 - 7g - 7i = 7g + 9i + 4 - 4g - 4i
\]
\[
2g + i + 7 = 3g + 5i + 4
\]
\[
3 - 4i = g
\]

Now focusing on:

\[
E[B] = E[E]
\]
\[
7g + 9i + 4(1 - g - i) = 4g + 7i + 8(1 - g - i)
\]
\[
7g + 9i + 4 - 4g - 4i = 4g + 7i + 8 - 8g - 8i
\]
\[
3g + 5i + 4 = -4g - i + 8
\]
\[
7g + 6i = 4
\]

Substituting we have:

\[
7g + 6i = 4
\]
\[
7(3 - 4i) + 6i = 4
\]
\[
21 - 28i + 6i = 4
\]
\[
17 = 22i
\]
\[
i = \frac{17}{22}
\]

Then we have:

\[
3 - 4i = g
\]
\[
3 - 4 \times \frac{17}{22} = g
\]
\[
\frac{66}{22} - \frac{68}{22} = g
\]
\[
\frac{-2}{22} = g
\]

At this point it looks like there is a problem because we are going to end up with: \( g = \frac{-2}{22}, i = \frac{17}{22}, \) and \( j = \frac{7}{22} \). This is a problem because \( g < 0 \). The question is, do these probabilities make player 1 indifferent among the pure strategies he plays with positive probabilities?

\[
E[A] = 9 \times \left( -\frac{2}{22} \right) + 8 \times \frac{17}{22} + 7 \times \frac{7}{22} = \frac{167}{22}
\]
\[
E[B] = 7 \times \left( -\frac{2}{22} \right) + 9 \times \frac{17}{22} + 4 \times \frac{7}{22} = \frac{167}{22}
\]
\[
E[E] = 4 \times \left( -\frac{2}{22} \right) + 7 \times \frac{17}{22} + 8 \times \frac{7}{22} = \frac{167}{22}
\]

4
They do – but because one of the probabilities violates the laws of probability, there cannot be an MSNE in the 3x3 game ... using all 3 strategies.\footnote{For the record, if you found player 1’s probabilities first you would have found that \(a = \frac{18}{113}, b = \frac{26}{113}, \) and \(e = \frac{69}{113}.\) These should lead to player 2 having an expected value of \(\frac{5622}{113}\) for each pure strategy he plays with positive probability.}

But there are actually two MSNE using only 2 strategies each from the 3x3 game. Consider the following 2x2 game:

<table>
<thead>
<tr>
<th></th>
<th>G</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>B</td>
<td>7, 8</td>
<td>9</td>
</tr>
</tbody>
</table>

Now for player 1 we have:

\[
\begin{align*}
E[A] &= 9g + 8(1-g) = 7g + 9(1-g) \\
9g + 8-8g &= 7g + 9 - 9g \\
g + 8 &= 9 - 2g \\
3g &= 1 \\
g &= \frac{1}{3}
\end{align*}
\]

So Player 2 would use G with probability \(\frac{1}{3}\) and I with probability \(\frac{2}{3}\). Now let’s look at ALL of Player 1’s expected values:

\[
\begin{align*}
E[A] &= 9 \cdot \frac{1}{3} + 8 \cdot \frac{2}{3} = \frac{25}{3} \\
E[B] &= 7 \cdot \frac{1}{3} + 9 \cdot \frac{2}{3} = \frac{25}{3} \\
E[C] &= 4 \cdot \frac{1}{3} + 4 \cdot \frac{2}{3} = \frac{12}{3} \\
E[D] &= 3 \cdot \frac{1}{3} + 6 \cdot \frac{2}{3} = \frac{15}{3} \\
E[E] &= 4 \cdot \frac{1}{3} + 7 \cdot \frac{2}{3} = \frac{18}{3}
\end{align*}
\]

Note that the strategies that Player 1 uses with positive probability, A and B, have an expected payoff of \(\frac{25}{3}\) when Player 2 uses the MSNE \((0, \frac{1}{3}, 0, \frac{2}{3}, 0)\), while the other strategies Player 1 could use have an expected payoff less than \(\frac{25}{3}\). Now we need to find Player 1’s probabilities using:

\[
\begin{align*}
E[G] &= E[I] \\
12a + 8(1-a) &= 0a + 11(1-a) \\
12a + 8 - 8a &= 11 - 11a \\
4a + 8 &= 11 - 11a \\
15a &= 3 \\
a &= \frac{3}{15} = \frac{1}{5}
\end{align*}
\]

So if Player 1 uses A with probability \(\frac{1}{5}\) and B with probability \(\frac{4}{5}\) this will make Player 2 indifferent over strategies G and I. Can Player 2 deviate to another pure strategy and receive a strictly higher
The answer is no, so there is an MSNE where: Player 1 chooses A with probability \( \frac{1}{5} \) and B with probability \( \frac{4}{5} \) while Player 2 chooses G with probability \( \frac{1}{3} \) and I with probability \( \frac{2}{3} \).

Now consider the 2x2 game:

<table>
<thead>
<tr>
<th></th>
<th>G</th>
<th>J</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>E</td>
<td>4.2</td>
<td>8</td>
</tr>
</tbody>
</table>

Setting:

\[
E[A] = E[E] \\
9g + 7(1 - g) = 4g + 8(1 - g) \\
9g + 7 - 7g = 4g + 8 - 8g \\
2g + 7 = 8 - 4g \\
6g = 1 \\
g = \frac{1}{6}
\]

So if Player 2 uses G with probability \( \frac{1}{6} \) and J with probability \( \frac{5}{6} \) then Player 1’s expected values (for all 5 pure strategies) are:

\[
E[A] = 9 \cdot \frac{1}{6} + 7 \cdot \frac{5}{6} = \frac{44}{6} \\
E[B] = 7 \cdot \frac{1}{6} + 4 \cdot \frac{5}{6} = \frac{27}{6} \\
E[C] = 4 \cdot \frac{1}{6} + 5 \cdot \frac{5}{6} = \frac{24}{6} \\
E[D] = 3 \cdot \frac{1}{6} + 1 \cdot \frac{5}{6} = \frac{8}{6} \\
E[E] = 4 \cdot \frac{1}{6} + 8 \cdot \frac{5}{6} = \frac{44}{6}
\]

so Player 1 would not want to switch to either B, C, or E. Now to find Player 1’s probabilities:

\[
E[G] = E[J] \\
12a + 2(1 - a) = 1a + 6(1 - a) \\
12a + 2 - 2a = 1a + 6 - 6a \\
10a + 2 = 6 - 5a \\
15a = 4 \\
a = \frac{4}{15}
\]
So if Player 1 uses A with probability $\frac{4}{15}$ and E with probability $\frac{11}{15}$ then Player 2’s expected values are:

$$E[F] = 11 \times \frac{4}{15} + 0 \times \frac{11}{15} = \frac{44}{15}$$
$$E[G] = 12 \times \frac{4}{15} + 2 \times \frac{11}{15} = \frac{70}{15}$$
$$E[H] = 1 \times \frac{4}{15} + 1 \times \frac{11}{15} = \frac{15}{15}$$
$$E[I] = 0 \times \frac{4}{15} + 4 \times \frac{11}{15} = \frac{44}{15}$$
$$E[J] = 1 \times \frac{4}{15} + 6 \times \frac{11}{15} = \frac{70}{15}$$

so that Player 2 would not want to switch to either F, H, or I. Thus, we have a second MSNE:

Player 1 chooses A with probability $\frac{4}{15}$ and E with probability $\frac{11}{15}$ while Player 2 chooses G with probability $\frac{1}{6}$ and J with probability $\frac{5}{6}$.

Now, let’s consider the last remaining 2x2:

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>J</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>9</td>
<td>11</td>
</tr>
<tr>
<td>E</td>
<td>4.5</td>
<td>8</td>
</tr>
</tbody>
</table>

Setting:

$$E[B] = E[E]$$
$$9i + 4(1 - i) = 7i + 8(1 - i)$$
$$9i + 4 - 4i = 7i + 8 - 8i$$
$$5i + 4 = 8 - i$$
$$6i = 4$$
$$i = \frac{2}{3}$$

Looking at Player 1’s expected values:

$$E[A] = 8 \times \frac{2}{3} + 7 \times \frac{1}{3} = \frac{23}{3}$$
$$E[B] = 9 \times \frac{2}{3} + 4 \times \frac{1}{3} = \frac{22}{3}$$
$$E[C] = 4 \times \frac{2}{3} + 5 \times \frac{1}{3} = \frac{18}{3}$$
$$E[D] = 6 \times \frac{2}{3} + 1 \times \frac{1}{3} = \frac{13}{3}$$
$$E[E] = 7 \times \frac{2}{3} + 8 \times \frac{1}{3} = \frac{22}{3}$$

Now, what do we notice here? We notice that $E[A] > E[B] = E[E]$. Thus, while Player 2’s choice of I with probability $\frac{2}{3}$ and J with probability $\frac{1}{3}$ makes Player 1 indifferent between B and E, it doesn’t really matter because Player 1 would choose A. This is the reason we do not have an MSNE for the 3x3 game. Thus, there is no MSNE for this 2x2 so there are 5 total Nash equilibria.

3 PSNE:
(1) P1 choose A, P2 choose G
(2) P1 choose B, P2 choose I
(3) P1 choose E, P2 choose J.
2 MSNE:

(4) Player 1 chooses A with probability $\frac{1}{5}$ and B with probability $\frac{4}{5}$ while Player 2 chooses G with probability $\frac{1}{3}$ and I with probability $\frac{2}{3}$.

(5) Player 1 chooses A with probability $\frac{4}{15}$ and E with probability $\frac{11}{15}$ while Player 2 chooses G with probability $\frac{1}{6}$ and J with probability $\frac{5}{6}$.

5. Two investors have each deposited $D$ with a bank. The bank has invested these deposits in a long-term project. If the bank is forced to liquidate its investment before the project matures, a total of $2r$ can be recovered, where $D > r > \frac{D}{2}$. If the bank allows the investment to reach maturity, however, the project will pay out a total of $2R$, with $R > D$.

There are 2 dates at which the investors can make withdrawals from the bank: date 1 is before the bank’s investment matures; date 2 is after. For simplicity, assume that there is no discounting. If both investors make withdrawals at date 1 then each receives $r$ and the game ends. If only one investor makes a withdrawal at date 1 then that investor receives $D$, the other receives $2r - D$, and the game ends. Finally, if neither investor makes a withdrawal at date 1 then the project matures and both investors make withdrawal decisions at date 2. If both investors make withdrawals at date 2 then each receives $R$ and the game ends. If only one investor makes a withdrawal at date 2 then that investor receives $2R - D$, the other receives $D$, and the game ends. If neither investor makes a withdrawal at date 2 then the bank returns $R$ to each investor and the game ends. Note that neither player observes the withdrawal decision of the other player at either date (in other words, at date 1 the players simultaneously choose to withdraw or not, and the same at date 2 – obviously once date 2 is reached both players know what the other player chose at date 1).

**a** Draw the extensive form version of this game.

**Answer:**
b There are 2 subgames in this game, one of which is the entire game and the other of which is the
game that begins at date 2. Write down the normal form version of the subgame that begins at
date 2.

Answer:
The normal form version of the subgame starting at date 2 is:

<table>
<thead>
<tr>
<th>Investor 1</th>
<th>Withdraw</th>
<th>Don’t Withdraw</th>
</tr>
</thead>
<tbody>
<tr>
<td>Investor 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Withdraw</td>
<td>$R, R$</td>
<td>$2R - D, D$</td>
</tr>
<tr>
<td>Don’t Withdraw</td>
<td>$D, 2R - D$</td>
<td>$R, R$</td>
</tr>
</tbody>
</table>

c Find the Nash equilibrium to the date 2 subgame in part b.

Answer:
The Nash equilibrium to the date 2 subgame in part b is that both investors choose Withdraw. Note that Withdraw is a strictly dominant strategy for both investors because \( R > D \).

d Find the subgame perfect Nash equilibria to this game.

**Answer:**

We know that in the date 2 subgame both investors will choose Withdraw because it is a strictly dominant strategy. Using this information we can create a 2x2 normal form game for the date 1 subgame, because the payoff when both players choose Don’t Withdraw is \( R, R \) (this follows from part e). The payoffs from any other choice of actions at date 1 is known because the game ends unless both investors choose Don’t Withdraw. The following normal form game results because the investors both choose Withdraw at date 2.

**Date 1 subgame**

<table>
<thead>
<tr>
<th>Investor 1</th>
<th>Withdraw</th>
<th>Don’t Withdraw</th>
</tr>
</thead>
<tbody>
<tr>
<td>Investor 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Withdraw</td>
<td>( r, r )</td>
<td>( D, 2r - D )</td>
</tr>
<tr>
<td>Don’t Withdraw</td>
<td>( 2r - D, D )</td>
<td>( R, R )</td>
</tr>
</tbody>
</table>

There are two (pure strategy) Nash equilibria to this game, one where both investors choose Withdraw and one where both investors choose Don’t Withdraw. In a sense it is like the Boxing-Opera game. Of course, the SPNE to the entire game is Investor 1 chooses Withdraw at date 1 and Withdraw at date 2 and Investor 2 chooses Withdraw at date 1 and Withdraw at date 2. Alternatively, another SPNE is that Investor 1 chooses Don’t Withdraw at date 1 and Withdraw at date 2 and Investor 2 chooses Don’t Withdraw at date 1 and Withdraw at date 2. Thus, there are 2 pure strategy SPNE to this game – one where both players run to the bank, and one where both players wait for the project to mature.

e If you are really bored you can write the strategic (or normal) form of the entire game and find all pure strategy NE.

**Answer:**

The strategic form will be a 4x4 game. Note that each player has two information sets and two actions at each information set, so four total strategies. I will list strategies as (first period action, second period action) for simplicity.

<table>
<thead>
<tr>
<th>Investor 2</th>
<th>( W, W )</th>
<th>( W, DW )</th>
<th>( DW, W )</th>
<th>( DW, DW )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Investor 1</td>
<td>( r, r )</td>
<td>( r, r )</td>
<td>( D, 2r - D )</td>
<td>( D, 2r - D )</td>
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<td>( 2r - D, D )</td>
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<td>( r, r )</td>
<td>( D, 2r - D )</td>
<td>( D, 2r - D )</td>
<td></td>
</tr>
</tbody>
</table>

The best responses are marked in the matrix. Recall that there were two SPNE: (1) \( W, W; W, W \) and \( DW, W; DW, W \). Note that there are three other PSNE – basically, both players would play Withdraw in the first stage and it does not matter what they do in the second stage because they never reach the second stage. Note that they cannot exploit the other player’s second stage strategy when both Withdraw in the first stage because they would need to switch to DW and would also need the other player to switch to DW.

6. Consider the following two-player sequential game. In period 1 each player is given $1. Player 1 has the opportunity to end the game right away by choosing Down and both players will get $1. However, Player 1 also have the option of sending the game to a second period by choosing Across, where $1 will be taken from Player 1 and $2 will be given to the Player 2. Player 2 then has the opportunity to end the game (with Player 1 getting nothing and Player 2 getting $3) by choosing Down or to send the game to a third period by choosing Across where $1 will be taken from him and $2 dollars will be given to Player 1. Player 1 can then end the game by choosing Down or continue the process by choosing Across. If the game gets to period 8 and Player 2 decides to send the game onto the next period by choosing Across, the game ends with both players getting $5.
a) Draw the extensive form version (game tree) of this game. Be sure to include all the components of the game in your diagram.

**Answer:**

![Game Tree Diagram]

I have added one extra feature in this game. Above each of the decision nodes (alternatively, because there is one decision node for each information set, above each information set) I placed a letter which identifies the decision node. This will facilitate describing the SPNE in part b.

b) Find the subgame perfect Nash equilibrium to this game.

**Answer:**

To solve this start from the smallest subgame (there are 8 subgames including the entire game) which is the one at decision node H. Player 2 should choose Down because $6 > $5. At decision node G Player 1 knows this and will then have to choose between Down ($4) and Across ($3) and will choose Down. This game then unravels back to the beginning with both players choosing Down whenever they get the chance. Thus, the SPNE is for Player 1 to choose Down at decision nodes A, C, E, and G and for Player 2 to choose Down at decision nodes B, D, F, and H.

c) What is the outcome if the subgame perfect Nash equilibrium is played?

**Answer:**

The outcome if the SPNE is played is both players receive $1 because the game ends instantly as Player 1 chooses Down at decision node A.

7. The citizens of Circleburg live in a city that is laid out in a perfect circle. The circumference of the circle is 12 miles. Residents live in houses which are distributed uniformly over the 12 miles. There are two competing gas stations, Chi Station and Epsilon Station. They are attempting to determine where to locate their respective stations. They know that residents of Circleburg will go to the gas station closest to their home. Assume that gas stations are concerned with maximizing the number of customers who visit their station. You may want to use a diagram to aid you when answering the questions. A pure strategy Nash Equilibrium (PSNE) for this game is a set of locations for the gas stations. Note that gas stations may locate at the same point on the circle.

a) Assuming that residents of Circleburg are able to drive clockwise and counterclockwise around the circle, describe the set of PSNE.
Answer:
When there are two competing gas stations any pair of points on the circle will be a PSNE. Regardless of where the stations locate they will each serve half of the customers.

b Delta Station has decided to enter the gas station market in Circleburg. Again, assume residents of Circleburg are able to drive clockwise and counterclockwise around the circle. If there is a PSNE to this game, find it. If there are multiple PSNE, describe the set of equilibria. If there are no PSNE, prove why there are none.

Answer:
The set of symmetric PSNE is when each station locates 4 miles apart from the other two stations. If you think about the city as the face of a clock, this would put one station at 12, one station at 4, and the other station at 8. Or one at 1, one at 5, and the other at 9, etc. Suppose that Chi is located at 12, Epsilon at 4, and Delta at 8. Each serves \( \frac{1}{2} \) of the customers at these locations. If Chi moves closer to Epsilon (say Chi moves to 3 o'clock) then Chi will keep the same amount of customers (but Delta would see an increase in customers and Epsilon a decrease). Thus, Chi cannot do any better by moving towards Epsilon. By a similar argument, Chi cannot do any better by moving towards Delta. The only other option for Chi is to locate between Epsilon and Delta, and this will only give Chi \( \frac{1}{4} \) of the customers in the market, so jumping between Epsilon and Delta makes Chi worse off. A similar analysis can be done for the other firms (and you all did this).

Note that all 3 stations locating at the same point is NOT a PSNE. This is because if this were the case then one station could move directly opposite from the others (if all were at 12 o’clock, one could move to 6 o’clock) and that station would now have \( \frac{1}{2} \) of the customers rather than \( \frac{1}{4} \). Actually, if one station moves ANYWHERE on the circle (and the other two stations remain at the same point) then the station that moves will have \( \frac{1}{2} \) of the customers (it’s like the 2 station case in the previous part).

We can find another set of PSNE when the stations are able to locate at the same points (call this set 2). Suppose 2 stations locate at the same point (12 o’clock) and one station locates directly opposite (6 o’clock). In this case, the station at 6 o’clock receives \( \frac{1}{2} \) of the customers and the other 2 stations each receive \( \frac{1}{4} \). Can any station do better? The station at 6 o’clock cannot do any better because it gets \( \frac{1}{2} \) of the customers regardless of where it locates on the circle if the other two are at the same point. Also, neither of the other stations can do better by independently changing their locations. Each station at the 12 o’clock location has \( \frac{1}{4} \) of the customers. If one of the 12 o’clock stations moves then it will still serve \( \frac{1}{4} \) of the customers. So no one station can move from these locations and make itself better off. Note that this is only a NE if the 2 stations are located directly opposite of the single station. If the 2 stations are located at 12 o’clock, and the single station is at 9 o’clock, then either of the stations at 12 o’clock could increase the number of customers it serves by locating at 8 o’clock. It would then serve over \( \frac{1}{4} \) of the customers (the actual number is \( \frac{9}{25} \)).

Finally, those are both special cases of the following result. Fix distinct locations for two stations (by distinct I mean not at the same point). So take 11 and 1 o’clock as an example. Draw a diameter through each of those points (the 11 o’clock diameter cuts the circle at 5 o’clock and the 1 o’clock diameter cuts the circle at 7 o’clock). Any location for the 3rd station between 5 and 7 o’clock will be a Nash. Essentially, the diameter through the 3rd station’s location will put the first 2 stations on different semicircles. To make this a little more formal, let \( |AB|_C \) be the distance between stations A and B on the part of the circle on which station C does not reside. Let \( |BC|_A \) be the distance between stations B and C on the part of the circle on which station A does not reside. Let \( |CA|_B \) be the distance between stations C and A on the part of the circle on which station B does not reside. Now let \( |AB|_C \geq |AB|_A \) and \( |BC|_A \geq |BC|_C \) and \( |CA|_B \geq |CA|_A \), then stations at those locations are part of a PSNE. For the symmetric case we have \( |AB|_C = |BC|_A = |CA|_B = 4 \) and \( |AB|_C = |BC|_A = |CA|_B = 8 \). Thus, the condition is met. For two stations (A and C) at 12 and one (B) at 6 we have \( |AB|_C = |BC|_A = 6 \) and \( |CA|_B = 12 \) so the condition is met. For a non-Nash case (stations A, B, and C located at 11, 12, and 1 o’clock respectively) we have \( |AB|_C = |BC|_A = 1 \)
and $|CA|_B = 10$. We have $|AB|_C = |BC|_A = 11$, which is great, but $|CA|_B = 2$, which violates our condition. To see that this is non-Nash note that station B could jump from its current side with $|CA|_B = 2$ to the side without be where $|CA|_{-B} = 10$ and gain a larger share of the market.

c The citizens of Circleburg have decided to outlaw backward thinking, which includes counterclockwise driving. The 3 gas stations are allowed to move their businesses from their previous locations. Now, residents of Circleburg will stop at the first gas station they see when they leave their home. If there is a PSNE to this game, find it. If there are multiple PSNE, describe the set of equilibria. If there are no PSNE, prove why there are none.

**Answer:**
There are no PSNE to this game. Essentially, each station wants to be “just counterclockwise” of the other stations. Consider the following 3 cases:

Case 1: All stations locate at the same point (say 12 o’clock). Any station has the incentive to locate at just before the 12 o’clock position to capture most of the market.

Case 2: Two stations locate at one point and a third locates at a different point. Again, either of the two stations at the same point has the incentive to move a little bit counterclockwise to take the entire part of the market it is currently sharing with the other firm. Also, the station at the different location has the incentive to move just counterclockwise of the two stations at the same location to take the whole market.

Case 3: All stations locate at different points. The best thing any one station can do is to locate a little bit counterclockwise of the station that has the most mass. If there is one station at 12 o’clock and one at 5 o’clock, then the 3rd station wants to locate just a little before 12 o’clock. But if that is the case then either of the other firms would like to locate just counterclockwise of the 3rd firm.

Essentially what happens is all the firms wish to play a game of leapfrog counterclockwise around the circle.