Problem Set 2 Answers
BPHD8110-001
February 13, 2018

1. Note that the unique Nash Equilibrium of the simultaneous Bertrand duopoly model with a continuous price space has each firm playing a weakly dominated strategy. Consider an alteration of the model in which prices must be named in some discrete unit of account (e.g. pennies) of size $\Delta$. Assume that both firms have identical constant marginal cost of $c$.

a Explain why the unique Nash Equilibrium of the Bertrand duopoly model with a continuous price space is a weakly dominated strategy.

Answer:
Consider charging some price $c + \varepsilon$ versus charging the NE price of $c$. Regardless of the price chosen by Firm 2, Firm 1 will receive a payoff of zero when choosing $p_1 = c$. If $p_2 < c$, then Firm 1 does not sell anything and receives zero. If $p_2 = c$, then the firms split the market but price equals cost. If $p_2 > c$, then Firm 1 captures the entire market but charges price equal to cost and earns zero. However, when $p_1 = c + \varepsilon$, for $\varepsilon > 0$, then there are some price choices by Firm 2 for which Firm 1 earns a positive profit. When $p_2 < c + \varepsilon$, then Firm 1 earns zero. But, when $p_2 \geq c + \varepsilon$, Firm 1 will earn a positive profit. It will either share the market at price $c + \varepsilon$ if $p_2 = c + \varepsilon$ or it will capture the entire market if $p_2 > c + \varepsilon$. Either way, Firm 1 earns a positive profit. Writing out two abbreviated rows of the normal form of the game we have:

<table>
<thead>
<tr>
<th>Firm 2</th>
<th>$p_2 &lt; c$</th>
<th>$p_2 = c$</th>
<th>$c &lt; p_2 &lt; c + \varepsilon$</th>
<th>$p_2 = c + \varepsilon$</th>
<th>$p_2 &gt; c + \varepsilon$</th>
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<tr>
<td>$p_1 = c$</td>
<td>$\pi_1 = 0$</td>
<td>$\pi_1 = 0$</td>
<td>$\pi_1 = 0$</td>
<td>$\pi_1 = 0$</td>
<td>$\pi_1 = 0$</td>
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<tr>
<td>$p_1 = c + \varepsilon$</td>
<td>$\pi_1 = 0$</td>
<td>$\pi_1 = 0$</td>
<td>$\pi_1 = 0$</td>
<td>$\pi_1 &gt; 0$</td>
<td>$\pi_1 &gt; 0$</td>
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This shows that choosing $p_1 = c$ is a weakly dominated strategy.

b Show that both firms naming prices equal to the smallest multiple of $\Delta$ that is strictly greater than $c$ is a PSNE of this game. Argue that it does not involve either firm playing a weakly dominated strategy.

Answer:
Assume that $c$ is a multiple of $\Delta$. Let $c + \Delta$ be the smallest price greater than $c$ that the firm can charge (if $c = \$5$, and $\Delta = \$0.01$, then $c + \Delta = \$5.01$). Suppose both firms charge $p_1 = p_2 = c + \Delta$, so that both firms earn a positive profit. Does either firm want to change its price? If either firm raises its price then it will earn zero. If either firm lowers its price then it must lower its price to $c$ (since choosing a price below $c$ results in negative profit), and will also earn zero. Thus, neither firm would choose to deviate from $p_1 = p_2 = c + \Delta$. In this scenario, it is also true that $p_1 = p_2 = c$ is also a PSNE of this game.

Now assume that $c$ is not a multiple of $\Delta$ (let $c = \$5.01$ and $\Delta = \$1$). In this case, $p_1 = p_2 = c$ is NOT a PSNE because choosing price equal to $c$ is not an available action. Let $p_1 = p_2 = \text{floor}_{\Delta}(c) + \Delta$, where $\text{floor}_{\Delta}(c)$ rounds $c$ down to the closest multiple of $\Delta$. In this case, choosing $p_1 = p_2 = \text{floor}_{\Delta}(c) + \Delta$ is a PSNE to the game, and both firms earn a positive profit in this equilibrium. To see this, if one firm lowers price then that firm will capture the entire market but will have $p_i < c$ and earn a negative profit. If one firm raises price then that firm sells nothing and earns zero profit. So neither firm wishes to deviate.
For completeness, note that it is possible to have a PSNE where both firms charge the same price and that price is greater than \( \text{floor}_\Delta(c) + \Delta \) (perhaps \( \text{floor}_\Delta(c) + 2\Delta \) or \( \text{floor}_\Delta(c) + 3\Delta \)). Consider the case where \( c = 5.01 \), \( Q = 100 - \frac{1}{2} (\min(p_1, p_2)) \) is the market demand function, and each firm’s demand follows the demand function from class (firm \( i \) sells 0 if \( p_i > p_j \), firm \( i \) sells \( \frac{1}{2}Q \) if \( p_i = p_j \), and firm \( i \) sells \( Q \) if \( p_i < p_j \)). Let \( \Delta = 2 \), so that prices must be even integers. Each firm charging \( p_1 = p_2 = 6 \) is a PSNE – note that neither firm wishes to deviate from that strategy given what the other firm is doing. Now, consider \( p_1 = p_2 = 8 \). This leads to a profit of \((8 - 5.01) * 48 = $143.52\) for each firm. Neither firm would wish to increase its price because then its profit would drop to 0. Would either firm be willing to cut its price to 6, the lowest possible price that yields nonnegative profit? If one firm dropped its price to 6 it would capture the entire market and sell 97 units at a profit of 99 cents apiece, leading to a profit of $96.03 for the firm. Since that is less than $143.52, neither firm would wish to deviate from \( p_1 = p_2 = 8 \). This result hinges upon the slope of the demand function and the size of \( \Delta \), as with the same structure in the example and \( \Delta = 1 \) the strategy profile \( p_1 = p_2 = 8 \) is NOT a PSNE (if one firm charges 7 then this firm makes $1.99 on 96.5 units and has a profit of $192.035).

Note that choosing a price of \( c + \Delta \) (or \( \text{floor}_\Delta(c) + \Delta \)) is not a weakly dominated strategy. Choosing a price of \( c + \Delta \) yields a strictly greater payoff than ANY other price choice when the other firm also chooses a price of \( c + \Delta \). The key difference between choosing \( c + \Delta \) when the price space is discrete and \( c \) when the price space is continuous is that choosing \( c \) when the price space is continuous does NOT yield a strictly higher payoff than other potential actions when the other firm chooses \( c \), it merely yields a payoff that is no worse than other payoffs.

c Argue that as \( \Delta \to 0 \), this equilibrium converges to both firms charging prices equal to \( c \).

**Answer:**

As \( \Delta \to 0 \) the price space moves from the discrete case to the continuous case. We have already seen how the PSNE when the price space is continuous is for both firms to charge a price of \( c \). We also know that both firms charging \( c + \Delta \) (or \( \text{floor}_\Delta(c) + \Delta \)) is a NE of the discrete price game. Taking \( \lim_{\Delta \to 0} c + \Delta \) we get \( c \).

2. Consider the three-player game below, which starts with P1:
a Show that the outcome \((A, A)\) that results in payoffs \((1, 1, 0)\) is NOT the outcome from a Nash equilibrium.

**Answer:**
Assume that \((A, A)\) is the outcome from a NE. We need to specify P3’s strategy in order to completely determine the NE. P3 can choose either \(L\) or \(R\). If P3 chooses \(L\), then P1 would choose \(D\). If P3 chooses \(R\), then P2 would choose \(D\). Regardless of what P3 chooses, either P1 or P2 would want to switch strategies.

b Find all pure strategy Nash equilibrium to this game.

**Answer:**
Suppose P3 choose \(L\). Then P1 would choose \(D\) (to receive 3) while P2 could choose either \(A\) or \(D\). Can any player switch strategies to receive a strictly higher payoff? No. P1 is receiving his highest payoff, P2 receives 0 regardless of whether \(A\) or \(D\) is chosen, and P3 receives 0 regardless of whether \(L\) or \(R\) is chosen. Thus, \(D, A, L\) and \(D, D, L\) are PSNE. Alternatively if P3 chooses \(R\) we have that \(D, D, R\) and \(A, D, R\) are PSNE.

Alternatively, consider what a 3-player strategic form of the game would be. It would be (in this case because all players have 2 strategies) a cube. It is difficult to visualize a cube, so we break it apart into two matrices, and let one player choose which matrix to play.

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<table>
<thead>
<tr>
<th></th>
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<th>D</th>
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<tbody>
<tr>
<td>A</td>
<td>1,1</td>
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</tr>
<tr>
<td>D</td>
<td>3,0</td>
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<td>D</td>
<td>0,3</td>
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3. Consider 2 firms who play a simultaneous Cournot game. Market demand is given by \( a - bq_1 - bq_2 \), with \( a > 0 \) and \( b > 0 \). Firm 1 has constant marginal cost of \( c_1 \) and Firm 2 has constant marginal cost of \( c_2 \) with \( c_1 < c_2 \) and no fixed costs for either firm. The profit to firm \( i \) is:

\[
\Pi_i (q_i, q_j) = (a - bq_i - bq_j) q_i - c_i q_i
\]

These firms, however, are not solely concerned with profit, but are also concerned with inequity in production. Thus, instead of maximizing profit they maximize utility by choosing quantity, where utility is given by:

\[
U_i (q_i, q_j) = \Pi_i (q_i, q_j) - \beta (q_i - q_j)^2
\]

where \( \beta > 0 \) is a constant which measures how much the firm dislikes inequity in production.

**a** Find the best response functions for this simultaneous Cournot game.

**Answer:**

Player 1 wants to maximize:

\[
U_1 (q_1, q_2) = (a - bq_1 - bq_2) q_1 - c_1 q_1 - \beta (q_1 - q_2)^2
\]

\[
\frac{\partial U_1}{\partial q_1} = a - 2bq_1 - bq_2 - c_1 - 2\beta (q_1 - q_2)
\]

\[0 = a - 2bq_1 - bq_2 - c_1 - 2\beta (q_1 - q_2)
\]

\[0 = a - 2bq_1 - bq_2 - c_1 - 2\beta q_1 + 2\beta q_2
\]

\[2bq_1 + 2\beta q_1 = a - bq_2 - c_1 + 2\beta q_2
\]

\[q_1 = \frac{a - bq_2 - c_1 + 2\beta q_2}{2b + 2\beta}
\]

Player 2 maximizes a similar equation:

\[
U_2 (q_1, q_2) = (a - bq_2 - bq_1) q_2 - c_2 q_2 - \beta (q_2 - q_1)^2
\]

\[
\frac{\partial U_2}{\partial q_2} = a - 2bq_2 - bq_1 - c_2 - 2\beta (q_2 - q_1)
\]

\[0 = a - 2bq_2 - bq_1 - c_2 - 2\beta (q_2 - q_1)
\]

\[0 = a - 2bq_2 - bq_1 - c_2 - 2\beta q_2 + 2\beta q_1
\]

\[2bq_2 + 2\beta q_2 = a - bq_1 - c_2 + 2\beta q_1
\]

\[q_2 = \frac{a - bq_1 - c_2 + 2\beta q_1}{2b + 2\beta}
\]

Technically, the best response functions are:

\[q_1 = \text{Max} \left[ 0, \frac{a - bq_2 - c_1 + 2\beta q_2}{2b + 2\beta} \right]
\]

\[q_2 = \text{Max} \left[ 0, \frac{a - bq_1 - c_2 + 2\beta q_1}{2b + 2\beta} \right]
\]

**b** Find the pure strategy Nash equilibrium for these firms which dislike inequity.

**Answer:**
To find this simply substitute in for either $q_1$ or $q_2$:

\[
\begin{align*}
(2b + 2\beta)q_1 &= a - c_1 + 2\beta q_2 - bq_2 \\
(2b + 2\beta)q_1 &= a - c_1 + (2\beta - b) q_2 \\
(2b + 2\beta)q_1 &= a - c_1 + (2\beta - b) \left( \frac{a - bq_1 - c_2 + 2\beta q_1}{2b + 2\beta} \right) \\
(2b + 2\beta)^2 q_1 &= 2ba + 2\beta a - 2bc_1 - 2\beta c_1 + (2\beta - b) (a - bq_1 - c_2 + 2\beta q_1) \\
(2b + 2\beta)^2 q_1 &= 2ba + 2\beta a - 2bc_1 - 2\beta c_1 + 2\beta a - 2\beta q_1 - 2\beta c_2 + 4\beta^2 q_1 - ba + b^2 q_1 \\
(2b + 2\beta)^2 q_1 + 4\beta b q_1 - 4\beta^2 q_1 - b^2 q_1 &= 2ba + 2\beta a - 2bc_1 - 2\beta c_1 + 2\beta a - 2\beta c_2 - ba + bc_2 \\
(2b + 2\beta)^2 q_1 + 4\beta b q_1 - 4\beta^2 q_1 - b^2 q_1 &= ba + 4\beta a - 2bc_1 - 2\beta c_1 - 2\beta c_2 + bc_2 \\
4\beta^2 q_1 + 8b\beta q_1 + 4\beta^2 q_1 + 4\beta b q_1 - 4\beta^2 q_1 - b^2 q_1 &= ba + 4\beta a - 2bc_1 - 2\beta c_1 - 2\beta c_2 + bc_2 \\
3b^2 q_1 + 12b\beta q_1 &= ba + 4\beta a - 2bc_1 - 2\beta c_1 - 2\beta c_2 + bc_2 \\
q_1 &= \frac{ba - 2bc_1 + bc_2}{3b^2 + 12b\beta} + \frac{4\beta a - 2\beta c_1 - 2\beta c_2}{3b^2 + 12b\beta} \\
q_1 &= \frac{a - 2c_1 + c_2}{3b + 12\beta} + \frac{4\beta a - 2\beta c_1 - 2\beta c_2}{3b^2 + 12b\beta}
\end{align*}
\]

Now you do not need the result in this form, it just makes it easy to see that if $\beta = 0$ the NE quantity matches that in part c below. For Firm 2 we will have a similar quantity, only replacing the $c_1$’s and $c_2$’s in Firm 1’s NE quantity with $c_2$’s and $c_1$’s. So:

\[
q_2 = \frac{a - 2c_2 + c_1}{3b + 12\beta} + \frac{4\beta a - 2\beta c_2 - 2\beta c_1}{3b^2 + 12b\beta}
\]

In the standard asymmetric cost Cournot model without inequity the PSNE is $q_i = \frac{a - 2c_i + c_j}{3b}$ for $i = 1, 2$. Let $q_i^*$ be Firm 1’s equilibrium production in the standard model and $\overline{q_i}$ be Firm 1’s equilibrium production in the model with inequity. Show that if $c_1 < c_2$ then $q_1^* > \overline{q_1}$.

**Answer:**

Now we need to show that:

\[
\begin{align*}
\frac{a - 2c_1 + c_2}{3b} &> \frac{ba + 4\beta a - 2bc_1 - 2\beta c_1 - 2\beta c_2 + bc_2}{3b^2 + 12b\beta} \\
(a - 2c_1 + c_2) (3b^2 + 12b\beta) &> (ba + 4\beta a - 2bc_1 - 2\beta c_1 - 2\beta c_2 + bc_2) 3b \\
3b^2 a - 6b^2 c_1 + 3b^2 c_2 + 12b\beta a - 24b\beta c_1 + 12b\beta c_2 &> 3b^2 a + 12b\beta a - 6b^2 c_1 - 6\beta bc_1 - 6\beta bc_2 + 3b^2 c_2 \\
-24b\beta c_1 + 12b\beta c_2 &> -6\beta bc_1 - 6\beta bc_2 \\
18\beta bc_2 &> 18\beta bc_1 \\
c_2 &> c_1
\end{align*}
\]

Technically I just showed that if $q_1^* > \overline{q_1}$ then $c_2 > c_1$, but we can simply reverse the steps to show that if $c_1 < c_2$ then $q_1^* > \overline{q_1}$.

4. Consider a capacity-constrained duopoly pricing game. Firm $j$’s capacity is $q_j$ for $j = 1, 2$, and each firm has the same constant cost per unit of output of $c \geq 0$ up to this capacity limit. Assume that the market demand function $x(p)$ is continuous and strictly decreasing at all $p$ such that $x(p) > 0$ and that there exists a price $\overline{p}$ such that $x(\overline{p}) = q_1 + q_2$. Suppose also that $x(p)$ is concave. Let $p(\cdot) = x^{-1}(\cdot)$ denote the inverse demand function.

Given a pair of prices charged, sales are determined as follows: consumers try to buy at the low-priced firm first. If demand exceeds this firm’s capacity, consumers are served in order of their valuations,
starting with high-valuation consumers. If prices are the same, demand is split evenly unless one firm’s
demand exceeds its capacity, in which case the extra demand spills over to the other firm. Formally,
the firms’ sales are given by the functions \( x_1(p_1, p_2) \) and \( x_2(p_1, p_2) \) satisfying:

\[
\begin{align*}
\text{If } p_j > p_i: & \quad \left\{ \begin{array}{l}
x_i(p_1, p_2) = \min \{ q_i, x(p_i) \} \\
x_j(p_1, p_2) = \min \{ q_j, \max \{ x(p_j) - q_i, 0 \} \}
\end{array} \right. \\
\text{If } p_2 = p_1 = p: & \quad \left\{ \begin{array}{l}
x_i(p_1, p_2) = \min \{ q_i, \max \{ x(p) - q_i, 0 \} \}
\end{array} \right.
\end{align*}
\]

a Suppose that \( q_1 < b_c(q_2) \) and \( q_2 < b_c(q_1) \), where \( b_c(\cdot) \) is the best-response function for a firm with
constant marginal costs of \( c \). Show that \( p_1^* = p_2^* = p(q_1 + q_2) \) is a Nash Equilibrium of this game.

**Answer:**
Consider that Firm 1 lowers its price, so that it chooses \( p_1 < p_1^* = p(q_1 + q_2) \). If Firm 1 does this,
then Firm 1 still sells \( q_1 \) units (since it was selling its capacity at \( p_1^* \)), but now sells all the units it can
for a lower price. This is not an optimal change by Firm 1.

Now consider that Firm 1 raises its price, so that it chooses \( p_1 > p_1^* = p(q_1 + q_2) \). We know that Firm
2 will produce \( q_2 \) units because it produces \( q_2 \) units when both Firm 1 and Firm 2 choose \( p(q_1 + q_2) \),
so it will produce \( q_2 \) units when it chooses \( p_2 = p(q_1 + q_2) \) and Firm 1 chooses \( p_1 > p(q_1 + q_2) \). Firm
1’s best response to Firm 2 producing \( q_2 \) is given by \( b_c(q_2) \), but by assumption this is more than Firm
1 can produce. The question then becomes what amount should Firm 1 produce if it cannot produce
\( b_c(q_2) \), and this amount is \( q_1 \). How does Firm 1 produce this amount? By choosing \( p_1 = p(q_1 + q_2) \).
This leads us right back to where we started. A similar pair of arguments can be made for Firm 2 to
show that \( p_1 = p_2 = p(q_1 + q_2) \) is a NE.

b Argue that if either \( q_1 > b_c(q_2) \) or \( q_2 > b_c(q_1) \), then no PSNE exists.

**Answer:**
This is only true if \( p(q_1 + q_2) > c \). If \( p(q_1 + q_2) > c \) then either firm can guarantee itself a positive
profit if it charges a low price (if \( p(q_1 + q_2) = 60 \) and \( c = 9 \), either firm (or both) can charge a price
of 10 and guarantee a positive profit for itself). Thus, any NE would have both firms making some
positive sales since they can guarantee a payoff better than 0 by charging a price close to \( c \).

Suppose \( p_1 < p_2 \). If Firm \( j \) is making positive sales, Firm \( i \) must be selling at capacity. Firm \( i \) can
then raise its price slightly and still sell at capacity. This will increase Firm \( i \)’s profit, meaning that
the two firms charging different prices cannot be a NE.

This leaves \( p_1 = p_2 \) for some level of \( p \) as the only potential NE of the game. There are 3 cases that
are possible:

a If \( p_1 = p_2 > p(q_1 + q_2) \), then at least one firm sells below capacity and this firm would be better off
by slightly lowering its price and either selling at full capacity or stealing all the customers from
the other firm.

b If \( p_1 = p_2 < p(q_1 + q_2) \), then both firms sell at full capacity. Each firm could gain by increasing its
price to \( p(q_1 + q_2) \) which would still enable it to sell at full capacity.

c If \( p_1 = p_2 = p(q_1 + q_2) \). This leads us back to the NE of the first part of this question. However,
now we have either \( q_1 > b_c(q_2) \) or \( q_2 > b_c(q_1) \) by assumption. Thus, the best response for Firm
1 to Firm 2 producing \( q_2 \) is to produce less than full capacity, which means that it needs to raise
its price. We have already seen that both firms charging different prices is not a NE, and that
both firms charging a price above \( p(q_1 + q_2) \) is not a NE.

5. Consider the following simultaneous game:
Player 2

<table>
<thead>
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<th></th>
<th>w</th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
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<td>5,3</td>
<td>6,2</td>
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<tr>
<td>Player 1</td>
<td>x</td>
<td>3,5</td>
<td>6,6</td>
<td>7,10</td>
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<tr>
<td>y</td>
<td>2,6</td>
<td>10,7</td>
<td>8,8</td>
<td>2,11</td>
</tr>
<tr>
<td>z</td>
<td>1,1</td>
<td>7,2</td>
<td>11,2</td>
<td>3,3</td>
</tr>
</tbody>
</table>

a Find all pure strategy Nash equilibria to the stage game.

Answer:
There are two PSNE to this game: (1) both choose $w$ and (2) both choose $z$.

For parts b-d, assume the game above is infinitely repeated.

b Find a strategy profile that results in an outcome path in which both players choose $x$ in every period and the strategy profile you have found is a subgame perfect Nash equilibrium (SPNE).

Answer:
Player 1 chooses $x$ in period 0 (or 1) and continues to choose $x$ unless a deviation he observes a deviation by player 2. Define a deviation as any choice by player 2 other than $x$. If a deviation occurs player 1 then chooses $z$ for the remainder of the game (note that there are two PSNE so two potential punishment strategies, choosing $w$ or choosing $z$). Player 2 uses the same strategy. If this is the set of strategies, they will be a SPNE if:

$$
\sum_{i=0}^{\infty} 6\delta^i \geq 10 + \sum_{i=1}^{\infty} 3\delta^i
$$

$$
\frac{6}{1-\delta} \geq 10 + \frac{3\delta}{1-\delta}
$$

$$
7\delta \geq 4
$$

$$
\delta \geq \frac{4}{7}
$$

Note that there are other strategy profiles that would also work – I’m sure you all have thought of some that I did not.

c Find a strategy profile that results in an outcome path in which both players choose $x$ in every odd period and $y$ in every even period and the strategy profile you have found is a subgame perfect Nash equilibrium.

Answer:
Players 1 and 2 choose $x$ in every odd period and $y$ in every even period unless a deviation occurs. A deviation occurs when anything but $x$ is chosen in an odd period and anything but $y$ is chosen in an even period. If a deviation occurs then the punishment will be to play $z$ every period for the remainder
of the game. This is an SPNE if:

$$
\sum_{i=0}^{\infty} 8 (\delta^2)^i + \sum_{i=0}^{\infty} 6\delta (\delta^2)^i \geq 11 + \sum_{i=1}^{\infty} 3\delta^i
$$

$$
\frac{8}{1-\delta^2} + \frac{6\delta}{1-\delta^2} \geq 11 + \frac{3\delta}{1-\delta}
$$

$$
\frac{8 + 6\delta}{(1-\delta)(1+\delta)} \geq 11 - \frac{3\delta}{1-\delta}
$$

$$
\frac{8 + 6\delta}{1+\delta} \geq 11 - 11\delta + 3\delta
$$

$$
\frac{8 + 6\delta}{1+\delta} \geq 11 - 8\delta
$$

$$
8 + 6\delta \geq 11 + 11\delta - 8\delta - 8\delta^2
$$

$$
8\delta^2 + 3\delta - 3 \geq 0
$$

You can solve this to show that if $\delta \geq 0.453$ then the players will prefer not to deviate. Also, we can check to make sure that no deviation occurs in the periods when $x$ is chosen. Then we would essentially have:

$$
\sum_{i=0}^{\infty} 6 (\delta^2)^i + \sum_{i=0}^{\infty} 8\delta (\delta^2)^i \geq 10 + \sum_{i=1}^{\infty} 3\delta^i
$$

$$
\frac{6}{1-\delta^2} + \frac{8\delta}{1-\delta^2} \geq 10 + \frac{3\delta}{1-\delta}
$$

$$
\frac{6 + 8\delta}{(1-\delta)(1+\delta)} \geq 10 + \frac{3\delta}{1-\delta}
$$

$$
\frac{6 + 8\delta}{1+\delta} \geq 10 - 10\delta + 3\delta
$$

$$
\frac{6 + 8\delta}{1+\delta} \geq 10 - 7\delta
$$

$$
6 + 8\delta \geq 10 + 10\delta - 7\delta - 7\delta^2
$$

$$
7\delta^2 + 5\delta - 4 \geq 0
$$

You can solve this to show that if $\delta \geq 0.479$ then the players will prefer not to deviate. Thus, the players would need a $\delta \geq 0.479$ in order for this set of strategies to be an SPNE. For $0.453 \leq \delta \leq 0.479$ the players would cooperate for the first period (assuming that the first period is 0 and is even) but then would deviate from the suggested strategies in the second period.

Again, there are other SPNE that can be used to support this as an outcome.

d Assume that $\delta = 0.4$, where $\delta$ is the discount factor. Find a strategy profile that results in an outcome path in which both players choose $y$ in every period and the strategy profile you have found is a subgame perfect Nash equilibrium.

**Answer:**

Similar to part b, but now both players choose $y$ instead of $x$ unless they observe a deviation. A deviation is any choice of strategy other than $y$ by the other player. If a deviation occurs, then the
other player will punish by choosing $z$ forever. This is an SPNE if:

$$
\sum_{i=0}^{\infty} 8\delta^i \geq 11 + \sum_{i=1}^{\infty} 3\delta^i \\
\frac{8}{1-\delta} \geq 11 + \frac{3\delta}{1-\delta} \\
8 \geq 11 - 11\delta + 3\delta \\
8\delta \geq 3 \\
\delta \geq \frac{3}{8}
$$

As long as $\delta \geq 0.375$ this will be an SPNE to the game. Since $\delta = 0.4$ it is an SPNE. What if the players had decided to use $w$ (the other PSNE to the game) as their punishment strategy? Then we would have:

$$
\sum_{i=0}^{\infty} 8\delta^i \geq 11 + \sum_{i=1}^{\infty} 4\delta^i \\
\frac{8}{1-\delta} \geq 11 + \frac{4\delta}{1-\delta} \\
8 \geq 11 - 11\delta + 4\delta \\
7\delta \geq 4 \\
\delta \geq \frac{4}{7}
$$

This would NOT be an SPNE to the game because $\delta = 0.4$ and we would need $\delta \geq 0.428571$ in order for it to be an SPNE. So the choice of punishment strategy makes a difference in this particular setting with $\delta = 0.4$.

6. Consider a developer who wishes to purchase $k$ parcels of land. If the developer purchases all $k$ parcels, the developer receives a payment of $D$. If the developer does not purchase all $k$ parcels, the developer receives a payment of $0$. The developer must purchase each parcel of land from the landowner who owns the land.

Consider $k$ landowners who each own a parcel of land. That parcel has value of $v_i$ to the landowner, where $v_i \sim U[0, D]$. The individual landowners know their own value for the land but the developer does not. Also, the landowners do NOT know the values of other landowners.

The game can be modeled as a sequential game. The developer makes an offer $w_i$ to each landowner. Each landowner only observes his own $w_i$ and must make a decision to accept or reject that $w_i$. If all $k$ landowners accept their own offer $w_i$, then the landowners each receive $w_i$ as a payment from the developer; the developer pays an amount $\sum_{i=1}^{k} w_i$, and the developer receives a payment of $D$. If ANY landowner chooses to reject $w_i$, then the developer makes no payment to any landowner and acquires no parcels of land – the developer receives $0$ but pays $0$. The landowners, who still own their land, receive $v_i$.

For simplicity, assume the seller sets $w_i = w_j$ for all $i, j$. The developer maximizes expected utility, and receives $D - kw$ if aggregation is successful (which occurs only if all $k$ landowners accept the offer) and $0$ if not. Note that $\Pr(\bar{\upsilon} > \upsilon)$ for the uniform distribution $U[0, \bar{D}]$ is $\frac{\bar{D}}{\bar{D}}$.

Assume the developer is risk neutral. Find a subgame perfect Nash equilibrium to this game with $k$ landowners. Be sure to set up the developer’s expected utility function correctly.

**Answer:**

Each of the landowners will accept any $w \geq v_i$ and reject any offers $w < v_i$. The developer, knowing
this, then maximizes expected utility by choosing \( w \):

\[
\max_w U_d = (D - kw) * \left( \frac{w}{D} \right)^k
\]

\[
\frac{\partial U_d}{\partial w} = \left( \frac{w}{D} \right)^k (-k) + (D - kw) k \left( \frac{w}{D} \right)^{k-1} \frac{1}{D}
\]

\[
0 = \left( \frac{w}{D} \right)^k (-k) + (D - kw) k \left( \frac{w}{D} \right)^{k-1} \frac{1}{D}
\]

\[
0 = (D - kw) \frac{1}{D} - k \frac{w}{D}
\]

\[
0 = (D - kw) - w
\]

\[
w = \frac{D}{k+1}
\]

The key to solving the developer’s problem is to correctly specify the probability that the offer is accepted. For one individual that probability is \( \frac{w}{D} \), while for all \( k \) individuals that probability is \( \left( \frac{w}{D} \right)^k \). So the SPNE is for all landowners to accept any \( w \geq v_i \) and to reject any \( w < v_i \), while the developer offers each landowner \( w = \frac{D}{k+1} \).

7. There are three rational pirates, A, B, and C, who find 100 gold coins and must decide how to distribute them. The pirates have a strict order of seniority: A is superior to B and C, and B is superior to C. The pirate world’s rules of distribution are as follows:

1. The most senior pirate proposes a distribution of coins among the pirates.
2. The pirates, including the proposer, then vote on whether to accept this distribution.
3. If the proposed allocation is approved by a majority or a tie vote, then the proposal is implemented.
4. If the proposed allocation is NOT approved, the proposer is thrown overboard from the pirate ship and left at sea, and the next most senior pirate makes a new proposal to begin the system again.

Pirates base their acceptance of the proposal on three factors. First, each pirate wants to remain on the ship (not be thrown overboard). Second, each pirate wants to maximize the number of gold coins he receives. Third, each pirate would prefer to throw another overboard, if all other results would otherwise be equal (so that if Pirate C is offered 0 coins he prefers to vote against the proposal as he gets to throw the proposer overboard if the proposal is not approved, and C is guaranteed to get at least 0 coins from any other proposal made).

a (5 points) Suppose that Pirate C is the only remaining pirate. How does he propose to distribute the coins? Hint: This is easy, do not overthink it.

Answer:
Pirate C proposes that he keeps all 100 coins because there are no other pirates with whom to share.

b (5 points) Now suppose that Pirate A has been thrown overboard and that Pirates B and C remain. How does Pirate B propose to distribute the coins?

Answer:
Pirate B proposes that he keeps all 100 coins because he (B) will vote to accept the distribution while C will vote to reject. But Pirate B has the tiebreaker so the distribution is accepted.
c (15 points) Now suppose all three pirates are still on the ship. Find a subgame perfect Nash equilibrium to this game.

**Answer:**

This is a little more involved. We can use parts a and b here as we know what Pirates B and C will do if they get to make a proposal. Now, look at the outcome if A is tossed overboard: Pirate B will receive all 100 coins and Pirate C will receive 0. If A offers B any amount of coins (even 100) will Pirate B accept the distribution – no, because if B votes against the distribution he knows he will get all the 100 coins PLUS he gets the satisfaction of throwing A overboard. So Pirate A will never offer Pirate B any coins. What about Pirate C? If A is thrown overboard C will get zero coins – Pirate A knows that if he offers C 0 coins C will vote against him and he (A) will be thrown overboard. But what if A offers C a single coin? Pirate C will accept because he knows that he will get 0 coins if A is thrown overboard – actually, Pirate C will accept any amount of coins above 0, but Pirate A will only offer 1 coin to C because A wants to maximize his own share of gold coins. So the SPNE is: Pirate A offers Pirate B 0 coins and Pirate C 1 coin (keeping 99 for himself). Pirate B rejects ANY offer by Pirate A and Pirate C votes to accept any offer by Pirate A in which C receives at least 1 coin. Pirate A votes for his own distribution because he wants to stay on the ship. If A is thrown overboard then Pirate B will propose 100 coins for himself and 0 for C. Pirate C will vote against B’s proposal and Pirate B will vote for his own proposals. If A and B are thrown overboard then Pirate C will propose that he keep all 100 coins and he will, obviously, vote for his own proposal. And then he will find another crew.