1. Nature determines whether Player 1 is strong or weak. Player 1 is strong with probability $\Pr(\text{strong})$ and weak with probability $\Pr(\text{weak})$. Player 1 is at a bar and has to decide to order beer or quiche. Player 2 is also at the bar and will decide whether or not to duel with Player 1. However, Player 2 cannot observe whether Player 1 is strong or weak, only whether beer or quiche is ordered. The structure of the game is that Nature determines Player 1’s type, Player 1 chooses whether to order beer or quiche, and then Player 2 observes this decision (but not the actual type) by Player 1 and chooses to duel or not duel Player 1.

If a weak Player 1 orders quiche and duels then the payoffs are $(1, 1)$ (Player 1 payoffs listed first and Player 2’s payoffs listed second); if he is not challenged to a duel then the payoffs are $(3, 0)$. If a weak Player 1 orders beer and duels then the payoffs are $(0, 1)$ and if not challenged to a duel the payoffs are $(2, 0)$. If a strong Player 1 orders quiche and is challenged to a duel then the payoffs are $(0, 0)$ and if not challenged to a duel the payoffs are $(2, 1)$. Finally, if a strong Player 1 orders beer and is challenged to a duel then the payoffs are $(1, 0)$ and if not the payoffs are $(3, 1)$. Suppose that $\Pr(\text{strong}) = 0.9$ and $\Pr(\text{weak}) = 0.1$.

a Write down the game tree for this game.

Answer:
b Find a pure strategy pooling perfect Bayesian equilibrium (PBE).

**Answer:**

Let’s choose the following equilibrium:

- \( P_{1W} \) choose beer
- \( P_{1S} \) choose beer \{P1 strategy
- \( \Pr (P_{1W}|\text{quiche}) = q \)
- \( \Pr (P_{1S}|\text{quiche}) = 1 - q \) \{P2 beliefs
- \( \Pr (P_{1W}|\text{beer}) = 0.1 \)
- \( \Pr (P_{1S}|\text{beer}) = 0.9 \)

??? if quiche is observed
- Not Duel if beer is observed \{P2 strategy

The reason Not Duel is chosen when beer is observed is that \( 0 \times 0.1 + 1 \times 0.9 = 0.9 > 1 \times 0.1 + 0 \times 0.9 = 0.1 \). The only thing left is to specify what happens if quiche were to be observed. In this case, Player 2 will choose Duel over Not Duel if:

\[
1 \times q + 0 \times (1 - q) \geq 0 \times q + 1 \times (1 - q)
\]

\[
q \geq 1 - q
\]

\[
q \geq \frac{1}{2}
\]
So there are 2 potential equilibria here:

\[
\begin{align*}
&P1_W \text{ choose beer} \quad \{P1 \text{ strategy} \\
&P1_S \text{ choose beer} \quad \{P1 \text{ strategy} \\
&\Pr(P1_W|\text{quiche}) = q \geq \frac{1}{2} \\
&\Pr(P1_S|\text{quiche}) = 1 - q < \frac{1}{2} \quad \{P2 \text{ beliefs} \\
&\Pr(P1_W|\text{beer}) = 0.1 \\
&\Pr(P1_S|\text{beer}) = 0.9 \\
&Duel \text{ if quiche is observed} \\
&Not \ Duel \ if \ beer \ is \ observed \quad \{P2 \text{ strategy} \\
\end{align*}
\]

and

\[
\begin{align*}
&P1_W \text{ choose beer} \quad \{P1 \text{ strategy} \\
&P1_S \text{ choose beer} \quad \{P1 \text{ strategy} \\
&\Pr(P1_W|\text{quiche}) = q \leq \frac{1}{2} \\
&\Pr(P1_S|\text{quiche}) = 1 - q < \frac{1}{2} \quad \{P2 \text{ beliefs} \\
&\Pr(P1_W|\text{beer}) = 0.1 \\
&\Pr(P1_S|\text{beer}) = 0.9 \\
&Not \ Duel \ if \ quiche \ is \ observed \\
&Not \ Duel \ if \ beer \ is \ observed \quad \{P2 \text{ strategy} \\
\end{align*}
\]

Note that only the first is a pooling equilibrium – in the 2\textsuperscript{nd} one the \(P1_W\) type would switch.

It is also possible that there is a pooling equilibrium where both \(P1_W\) and \(P1_S\) choose quiche. If this were to happen we would need:

\[
\begin{align*}
&P1_W \text{ choose quiche} \quad \{P1 \text{ strategy} \\
&P1_S \text{ choose quiche} \quad \{P1 \text{ strategy} \\
&\Pr(P1_W|\text{quiche}) = 0.1 \\
&\Pr(P1_S|\text{quiche}) = 0.9 \\
&\Pr(P1_W|\text{beer}) = q \geq \frac{1}{2} \quad \{P2 \text{ beliefs} \\
&\Pr(P1_S|\text{beer}) = 1 - q < \frac{1}{2} \\
&Not \ Duel \ if \ quiche \ is \ observed \\
&Duel \ if \ beer \ is \ observed \quad \{P2 \text{ strategy} \\
\end{align*}
\]

Note that I have skipped the derivations here and simply stated the equilibrium which is one. Thus, \(P2\) must believe that there is a greater than 50% chance that a \(P1_W\) is choosing beer in order for this to be an equilibrium. The potential equilibrium where Not Duel is chosen regardless of the beer/quiche decision is NOT an equilibrium because then \(P1_S\) would switch to beer.

\textbf{c} Is there a pure strategy separating PBE? Explain why or why not.

\textbf{Answer:}

Start with the obvious one:

\[
\begin{align*}
&P1_W \text{ choose quiche} \quad \{P1 \text{ strategy} \\
&P1_S \text{ choose beer} \quad \{P1 \text{ strategy} \\
&\Pr(P1_W|\text{quiche}) = 1 \\
&\Pr(P1_S|\text{quiche}) = 0 \\
&\Pr(P1_W|\text{beer}) = 0 \quad \{P2 \text{ beliefs} \\
&\Pr(P1_S|\text{beer}) = 1 \\
&Not \ Duel \ if \ quiche \ is \ observed \\
&Duel \ if \ beer \ is \ observed \quad \{P2 \text{ strategy} \\
\end{align*}
\]
Note that this is NOT an equilibrium to the game because the $P_1W$ type would switch to beer to receive a payoff of 2 (since Not Duel is chosen when beer is observed) as opposed to 1 from this equilibrium. The other alternative is:

- $P_1W$ choose quiche
- $P_1S$ choose beer

\[
\begin{align*}
\Pr (P_1W|\text{quiche}) &= 0 \\
\Pr (P_1S|\text{quiche}) &= 1 \\
\Pr (P_1W|\text{beer}) &= 1 \\
\Pr (P_1S|\text{beer}) &= 0
\end{align*}
\]

\text{Duel if quiche is observed} \quad \text{Not Duel if beer is observed}

Once again the $P_1W$ type would switch since it would receive 3 from quiche and 0 from beer in this potential equilibrium. So there are no pure strategy separating equilibria in this game. The general logic is that if $P_2$ can tell which type player 1 is (which $P_2$ can in a separating equilibrium), then $P_2$ will choose to Duel the weak type and Not Duel the strong type, regardless of what they choose. The weak type would then wish to deviate by masquerading as a strong type and choosing whichever action the strong type chooses (which is why we have pooling equilibria for both types choosing Quiche and both types choosing Beer).

2. Consider the following game:

\begin{center}
\begin{tabular}{c|ccc}
& L & M & R \\ 
L' & 1 & 3 & 4 \\ 
M' & 1 & 2 & 4 \\ 
R' & 4 & 0 & 2 \\ 
\end{tabular}
\end{center}

\textbf{a} Write down the normal form (or matrix) for this game.

\textbf{Answer:}

The normal form of this game is:

\begin{tabular}{c|ccc}
& L & M & R \\ 
L & 1,3 & 1,2 & 4,0 \\ 
M & 4,0 & 0,2 & 3,3 \\ 
R & 2,4 & 2,4 & 2,4 \\ 
\end{tabular}

\textbf{b} Find all pure strategy Nash equilibria (PSNE), subgame perfect Nash equilibria (SPNE), and perfect Bayesian equilibria (PBE) in this game.
Answer:

First, note that there is only one subgame to this game (the entire game), so that all the PSNE and SPNE will be the same. Using the matrix we see:

\[
\begin{array}{ccc}
\text{Player 2} & L' & M' & R' \\
\text{Player 1} & \begin{array}{ccc}
L & 1 & 2 & 4 \\
M & 0 & 2 & 3 \\
R & 2 & 4 & 2 \\
\end{array}
\end{array}
\]

there is only one PSNE (and so only one SPNE as well), which is for Player 1 to choose R and Player 2 to choose M'. Also, Player 1 choose R and Player 2 choose M' is the ONLY PSNE to this game. So it will be the only PBE as well, only we need to find the beliefs (probabilities) such that Player 2 would choose M'.

To find the beliefs, let \( q \) be Player 2's belief that Player 1 chooses L and \( (1 - q) \) be Player 2's belief that Player 1 chooses M. Player 2's expected value of each strategy is:

\[
\begin{align*}
E[L'] &= 3q + 0 \cdot (1 - q) = 3q \\
E[M'] &= 2q + 2 \cdot (1 - q) = 2 \\
E[R'] &= 0q + 3 \cdot (1 - q) = 3 - 3q
\end{align*}
\]

We can tell that Player 2 will choose \( L' \) if:

\[
3q > 2 \\
3q > 3 - 3q
\]

or

\[
q > \frac{2}{3} \\
q > \frac{1}{2}
\]

We need both to be true, so if \( q > \frac{2}{3} \) Player 2 will choose \( L' \). Player 2 will choose \( M' \) if:

\[
2 > 3q \\
2 > 3 - 3q
\]

or

\[
q < \frac{2}{3} \\
q > \frac{1}{3}
\]

So if \( q \in \left(\frac{1}{3}, \frac{2}{3}\right) \) Player 2 will choose \( M' \). Finally, Player 2 will choose \( R' \) if:

\[
3 - 3q > 3q \\
3 - 3q > 2
\]

or

\[
q < \frac{1}{2} \\
q < \frac{1}{3}
\]

Again, we need both to be true, so if \( q < \frac{1}{3} \) Player 2 will choose \( R' \).

Thus, there is a class of PBE for this game where Player 1 chooses R, Player 2 believes that Player 1 has chosen L with some probability between \( \left(\frac{1}{3}, \frac{2}{3}\right) \), and Player 2 chooses \( M' \).
3. Consider the following Sender-Receiver game that has been slightly modified as type $t_2$ now has a third option, $M$, which ends the game without allowing the Receiver a chance to make a decision. Note that the probability of being a sender type $t_1$ is $\frac{1}{4}$ and the probability of being a sender type $t_2$ is $\frac{3}{4}$.

a) Find all pure strategy pooling perfect Bayesian equilibria.

**Answer:**
There are two potential pooling equilibria, $t_1$ and $t_2$ choose $L$ or $t_1$ and $t_2$ choose $R$. Hopefully the first thing you noticed is that $t_2$ will NEVER choose $L$ because $t_2$ is always guaranteed to be better off by choosing $M$. Thus, the only pooling equilibrium that needs to be considered is where both $t_1$ and $t_2$ choose $R$.

If $t_1$ and $t_2$ choose $R$ then the Receiver believes:

\[
\begin{align*}
\Pr(t_1|R) &= 0.25 \\
\Pr(t_2|R) &= 0.75 \\
\Pr(t_1|L) &= q \\
\Pr(t_2|L) &= 1 - q
\end{align*}
\]
If $R$ is observed the Receiver’s expected value from $U$ is:

$$E[U|R] = \frac{1}{4} \times 4 + \frac{3}{4} \times 4 = \frac{16}{4}$$

The Receiver’s expected value from $D$ is:

$$E[D|R] = \frac{1}{4} \times 6 + \frac{3}{4} \times 2 = \frac{12}{4}$$

So the Receiver will choose $U$ if $R$ is observed because $\frac{16}{4} > \frac{12}{4}$.

Now, when will the Receiver choose $U$ if $L$ is observed (there are two ways to think about this – we will use the $q$ and $1 - q$ first).

$$E[U|L] = 6q + 5(1 - q) = 6q + 5 - 5q = q + 5$$
$$E[D|L] = 2q + 2(1 - q) = 2$$

As long as $q + 5 > 2$ or $q > -3$ the Receiver will choose $U$ if $L$. This should make sense because if the Receiver chooses $D$ if $L$ he gets 2 regardless of which type chooses $L$, and if he chooses $U$ he gets at least 5 regardless of which type chooses $L$. So the Receiver will always choose $U$ if $L$.

Now, will either $t_1$ or $t_2$ switch? Type $t_1$ receives 9 from choosing $R$ and would receive 4 from switching to $L$ so $t_1$ will not switch. Type $t_2$ receives 8 from choosing $R$, would receive 6 from switching to $M$, and would receive 4 from switching to $L$, so $t_2$ will not switch. Thus we have a pure strategy pooling equilibrium.

```
t_1 choose R
t_2 choose R
```
Sender’s strategy

```
Pr(t_1|R) = 0.25
Pr(t_2|R) = 0.75
Pr(t_1|L) = q \in [0, 1]
Pr(t_2|L) = 1 - q
```
Receiver’s beliefs

```
U if R
U if L
```
Receiver’s strategy

Note that this is a WPBE. If we wanted a strong pooling PBE then we would have $Pr(t_1|L) = 1$ and $Pr(t_2|L) = 0$ because the Receiver should know that type $t_2$ will never choose $L$.

**b** Find all pure strategy separating perfect Bayesian equilibria.

**Answer:**

There are four potential separating equilibria here:

1. $t_1$ choose $L$, $t_2$ choose $R$
2. $t_1$ choose $L$, $t_2$ choose $M$
3. $t_1$ choose $R$, $t_2$ choose $L$
4. $t_1$ choose $R$, $t_2$ choose $M$

We can rule out (3) because $t_2$ will never choose $L$.

Also, recall from part a that the Receiver will always choose $U$ if $L$ regardless of who chooses $L$.

Starting with (1), if $t_1$ chooses $L$ the Receiver chooses $U$ and if $t_2$ chooses $R$ the Receiver will also choose $U$. Now, $t_1$ will switch to $R$ because $9 > 4$ so this is not an equilibrium.

Jumping to (4), if $t_1$ chooses $R$ the Receiver chooses $D$ and even though $L$ is not chosen by any Sender we know the Receiver will chooses $U$ if $L$ is observed. However, this means that the $t_1$ type will switch to $L$ because $t_1$ will receive 4 from choosing $L$ and only receives 2 from choosing $R$. So (4) is not a separating equilibrium.
That leaves (2).

We know that if \( t_1 \) choose \( L \) the Receiver will choose \( U \). All that we need to do now is find the probabilities that lead to the Receiver choosing \( U \) or \( D \) when \( R \) is chosen because the Receiver never observes \( R \) in equilibrium (2). But if we think carefully, if the Receiver chooses \( U \) if \( R \) then both \( t_1 \) and \( t_2 \) will switch to \( R \) because both Sender types’ highest payoffs are when they choose \( R \) and the Receiver chooses \( U \). This just leaves us to find the beliefs that the Receiver uses to choose \( D \) if \( R \).

\[
\begin{align*}
E[ U | R ] &= 4q + 4(1-q) = 4 \\
E[ D | R ] &= 6q + 2(1-q) = 6q + 2 - 2q = 4q + 2
\end{align*}
\]

so

\[
4q + 2 \geq 4q \geq 1/2
\]

So the only separating equilibrium occurs when:

\[
\begin{array}{c}
t_1 \text{ choose } \ L \\
t_2 \text{ choose } \ M
\end{array}
\]

Sender’s strategy

\[
\begin{array}{c}
\Pr(t_1 | R) = q \geq 1/2 \\
\Pr(t_2 | R) = 1 - q \\
\Pr(t_1 | L) = 1 \\
\Pr(t_2 | L) = 0
\end{array}
\]

Receiver’s beliefs

\[
\begin{array}{c}
D \text{ if } R \\
U \text{ if } L
\end{array}
\]

Receiver’s strategy

4. In the Battle of the Sexes game there are 2 players who are simultaneously deciding whether or not to attend a boxing match or an opera performance. However, now both players are uncertain about the value that the other player places on his or her favorite event. Thus, the normal form is:

<table>
<thead>
<tr>
<th></th>
<th>Opera</th>
<th>Boxing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player A</td>
<td>Opera</td>
<td>2 + ( t_A ), 1</td>
</tr>
<tr>
<td></td>
<td>Boxing</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

where \( t_A \) is privately known by Player A and \( t_B \) is privately known by Player B. Both \( t_A \) and \( t_B \) are independent draws on a uniform distribution from \([0, x]\).

Let \( A \) be the critical value such that if \( t_A \geq A \) then Player A plays Opera and if \( t_A < A \) then Player A plays Boxing. Let \( B \) be the critical value such that if \( t_B \geq B \) then Player B plays Boxing and if \( t_B < B \) then Player B plays Opera. In this equilibrium, Player A plays Opera with probability \( \frac{x-A}{x} \) and Player B plays Boxing with probability \( \frac{x-B}{x} \).

a Find the critical values \( A \) and \( B \). They will be a function of \( x \) (recall that \( x \) is the upper bound of the uniform distribution).

**Answer:**

Player A should play Opera if \( E_A \{ \text{Opera} \} \geq E_A \{ \text{Boxing} \} \). We have:

\[
\begin{align*}
E_A \{ \text{Opera} \} &= \left( \frac{B}{x} \right) \times (2 + t_A) + \left( \frac{x - B}{x} \right) \times 0 \\
E_A \{ \text{Oper}a \} &= \left( \frac{B}{x} \right) \times (2 + t_A)
\end{align*}
\]
Now, $E_A[Boxing]$ is:

$$E_A[Boxing] = \left(\frac{x-B}{x}\right) * (1) + \left(\frac{B}{x}\right) * 0$$

$$E_A[Boxing] = \left(\frac{x-B}{x}\right) * (1)$$

So we need:

$$\left(\frac{B}{x}\right) * (2 + t_A) \geq \left(\frac{x-B}{x}\right)$$

For Player B, Player B should play Boxing if $E_B[Boxing] \geq E_B[Opera]$. We have:

$$E_B[Boxing] = \frac{A}{x} * (2 + t_B) + \left(\frac{x-A}{x}\right) * 0$$

$$E_B[Boxing] = \frac{A}{x} * (2 + t_B)$$

Now $E_B[Opera]$ is:

$$E_B[Opera] = \left(\frac{x-A}{x}\right) * 1 + \left(\frac{A}{x}\right) * 0$$

$$E_B[Opera] = \left(\frac{x-A}{x}\right) * 1$$

So we need:

$$\frac{A}{x} * (2 + t_B) \geq \left(\frac{x-A}{x}\right)$$

We now have 2 equations and 2 unknowns:

$$\left(\frac{B}{x}\right) * (2 + t_A) \geq \left(\frac{x-B}{x}\right)$$

$$\frac{A}{x} * (2 + t_B) \geq \left(\frac{x-A}{x}\right)$$

The key is to realize that $A$ and $B$ are just some signal $t_A^*$ and $t_B^*$ respectively. Also, we can remove the inequalities because we are trying to find the single critical value, so we now have:

$$\frac{t_B^*}{x} * (2 + t_A^*) = \left(\frac{x-t_B^*}{x}\right)$$

$$\frac{t_A^*}{x} * (2 + t_B^*) = \left(\frac{x-t_A^*}{x}\right)$$

Now we just need to solve for $t_A^*$ and $t_B^*$.

$$2 + t_A^* = \frac{x-t_B^*}{t_A^*}$$

$$2 + t_B^* = \frac{x-t_A^*}{t_B^*}$$

Or:

$$t_A^* = \frac{x-t_B^*}{t_A^*} - 2$$

$$t_A^* = \frac{x-t_B^*}{t_A^*} - \frac{2t_B^*}{t_B^*}$$

$$t_A^* = \frac{x-3t_B^*}{t_B^*}$$
By the same method we get:

\[ t_B^* = \frac{x - 3t^*_A}{t^*_A} \]

And now we just plug in \( t_A^* \):

\[ t_B^* = \frac{x - 3\left(\frac{x - 3t^*_A}{t^*_B}\right)}{x - 3t^*_B} \]

Now it’s just some algebra:

\[ t_B^* = \frac{x(t_B^* - 3x + 9t_B^*)}{x - 3t_B^*} \]

\[ t_B^* = \frac{xt_B^* - 3x + 9t_B^*}{x - 3t_B^*} \]

\[ t_B^* x - 3(t_B^*)^2 = xt_B^* - 3x + 9t_B^* \]

\[ -3(t_B^*)^2 = -3x + 9t_B^* \]

\[ (t_B^*)^2 = x - 3t_B^* \]

\[ (t_B^*)^2 + 3t_B^* - x = 0 \]

Now we can just use the quadratic equation to find:

\[ t_B^* = \frac{-3 \pm \sqrt{9 - 4 \cdot 1 \cdot (-x)}}{2} \]

\[ t_B^* = \frac{-3 \pm \sqrt{9 + 4x}}{2} \]

The last piece to realize is that we need \( t_B^* \geq 0 \) since \( t_B^* \in [0, x] \), so:

\[ t_B^* = \frac{-3 + \sqrt{9 + 4x}}{2} \]

We can then plug this in to find \( t_A^* \), or we can just state “by symmetry” because the equations for \( t_A^* \) and \( t_B^* \) are the same to find:

\[ t_A^* = \frac{-3 + \sqrt{9 + 4x}}{2} \]

b Explain how the proposed equilibrium satisfies the criteria for a Bayes-Nash equilibrium.

**Answer:**

In a Bayes-Nash equilibrium, each player needs to specify an action for each potential type he could be. Look at the equilibrium as it is described. Consider Player A. For any signal \( t_A < A \), Player A plays boxing, and for any signal \( t_A \geq A \) Player A plays opera. For Player B, for any \( t_B \geq B \) Player B plays Boxing and for any \( t_B < B \) Player B plays Opera. Thus, an action is specified for any potential type for both players. Part a of this problem shows how both are using best responses to the others choice of strategy.