Evolution of Heterogeneous Beliefs and Asset Overvaluation.

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Abstract

I analyze a model in which different agents have different non-rational expectations about the future price and cash flows of a risky asset. The beliefs in the society evolve according to a very general class of evolution functions that are monotone; that is if one type has increased its share in the population then all types with higher profit should also have increased their shares. I show that the price of the risky asset converges to the risk-neutral fundamental price even though all agents in the economy are risk-averse. The risky asset thus becomes overvalued as compared to the equilibrium with rational expectations. The overvaluation is a result of the evolution of beliefs and does not rely on such asymmetric assumptions as short-sale constraints or optimistic bias.

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1 Introduction

The nature of asset market behavior in the presence of heterogenous agents has been debated in the literature since as early as the nineteen fifties. In 1953, Milton Friedman argued that economists should restrict their attention to models where all agents have rational expectations. His argument was that even if there were irrational (in terms of beliefs) traders on the market, they would be making consistently less money than rational traders.
and thus eventually would be driven out of the market, and their price impact would become negligible.

Despite the intuitive appeal of this argument, over the past years researchers have accumulated a large body of evidence that shows that heterogeneity of beliefs can be persistent. In particular, it has been shown that under some reasonable conditions such as incomplete markets (Blume and Easley (1992), (2006)) the market selection hypothesis can fail. Indeed, it is possible that traders with rational expectations will be driven out of the market and their consumption path will converge to zero. DeLong, Shleifer, Summers and Waldmann (1990) (DSSW) argue that the noise traders with optimistic beliefs can make higher profit than the agents with rational expectations. The reason is that optimists have higher demand for risky assets, and thus as long as the assets have positive excess return, optimists will be making more money.

The model in this paper is built upon the intuition in DSSW. I investigate asset market behavior under two assumptions. First, I assume that different types of agents have different beliefs. Second, I assume that the types who earn higher profit increase their market weight. Unlike DSSW, I consider the model where all agents have non-rational expectations. This enables me to completely characterize the equilibrium dynamic, whereas in the DSSW only a single limiting case was considered.

The setting without rational expectations is used by many authors. See, for example, Brock and Hommes (1997), (1998), Brock et al. (2005), Chiarella and He (2001), (2002), Levy and Levy (1996) Lux and Marchesi (2000), Hommes et al. (2005), etc. Similar to this literature, I consider the following framework. I assume that there are two assets: a riskless bond available in perfectly elastic supply, and a risky asset that pays random dividends and has a fixed supply. Agents are different in their beliefs about future price and dividend distributions of the risky asset. Since different agents have different beliefs, they will make different portfolio choices and will earn different profits. The weights of different types evolve in such a way that agents with higher profit gain higher market share. We can think of this as a market selection mechanism, similar to that suggested by Friedman.

The major difference of my paper from this literature is in the way I model the selection mechanism. The literature typically models the mechanism at the level of individual forecasting rules. In other words, it is usually assumed that there are two or three types of agents that follow particular rules (e.g. trend extrapolation, trend reversion, fundamentalists) to update their beliefs. The selection mechanism favors those rules that perform better. The main difficulty with this approach is that, one has to specify the exact rules that agents use to update their beliefs. However, there are many different rules that agents could use to update their beliefs. A priori, it is not clear which combination of rules would be the best to model the stock market, and how robust the results are to different choices of rules.
Another problem with this approach is analytical tractability. The dynamic with three rules is already difficult to analyze analytically. Considering more than three rules makes a formal analysis almost impossible.

Instead of modeling the selection mechanism at the level of individual forecasting rules, I model it at the level of the aggregated distribution of beliefs. This means that the selection mechanism specifies how a distribution of beliefs evolve in the economy. There are no direct restrictions placed on how individuals update their forecasts. Effectively individuals can use any update technique as long as the change in aggregated distribution of beliefs satisfies few natural properties such as monotonicity.

The main properties that I impose on the evolution process are monotonicity, slow-speed and non-triviality. Monotonicity is a requirement that makes the selection mechanism to reflect Friedman’s idea. It says that if some type has increased its market weight, then all types with higher profit should increase their market weights as well. The slow-speed assumption requires that the market weights of types change sufficiently slowly. Non-triviality states that if agents make different profits, then their weights should change.

In the paper, I am mostly interested in the analysis of the deterministic skeleton of the dynamic which is when the selection mechanism depends on conditional expectation of the wealth. The main result of the paper is as follows. Assume that a risk-neutral fundamental price can be a market-clearing price for some beliefs’ distribution. Then for any Lipschitz continuous selection mechanism that satisfies weak-monotonicity, non-triviality and slow speed properties, the asset price will converge to the risk-neutral fundamental price even though all market participants are risk-averse. Furthermore, I show that even in the long-run heterogeneity of beliefs will persist. This means that, first, in the long-run, the asset will be overvalued as compared to the equilibrium with rational expectations. Second, it means that the Friedman selection mechanism does not bring the economy to the correct prices and beliefs as one would expect. Instead the asset price ends up being higher than the correct one, and agents with different beliefs (including incorrect ones) will co-exist.

To show the robustness of this result, I analyze the case when the selection mechanism depends on realized wealth. I show that if evolution depends on the actually realized profits then with probability one the equilibrium price will be absorbed by a symmetric interval around the risk-neutral price. Thus, it is still the risk-neutral price that drives the price dynamic and not the ("correct") risk-adjusted price.

The basic intuition behind the paper’s main result is similar to DSSW. Optimists mistakenly find the risky asset to be more attractive than it actually is and consequently hold more units of the asset. It is sub-optimal in that they end up with a portfolio that is too risky. However, the selection mechanism does not care about the risk taken except insofar as it affects the wealth made. Because of the risk premium, optimists on average make more
money, thus their market share increases and it pushes the asset price up.

Even though my model predicts overvaluation, the result does not rely on built-in asymmetric assumptions such as an optimistic bias or short-sale constraints. For example, Miller (1977) showed that short-sale constraints together with heterogenous beliefs can generate asset overvaluation. The idea is simple: the asset can get overvalued because of optimistic traders and because of short-sale constraints the mispricing will not necessarily get corrected. Thus, in the Miller’s world there is an asymmetry between optimists and pessimists. The former have no restrictions in their ability to drive the price up, whereas the latter can affect the price only as much as short-sale constraints allow. While short-sale constraints are real I show that they are not necessary to achieve overvaluation. I consider the model without short-sale constraints which puts both optimists and pessimists in a symmetric situation. Nonetheless, it is still the case that the risky asset ends up to be overvalued.

Recently in the literature, many explanations of overvaluation have been offered using both rational and behavioral models. Rational theories explain it by increasing opportunities for portfolio diversification (see Merton (1987) and Heaton and Lucas (2000)) or by the decline in macroeconomic risk in the US economy (see Lettau et al. (2004)). More behavioral explanations follow Miller’s idea, which I discussed above. They assume that, for example due to overconfidence, investors have different opinions, which together with short-sale constraints make assets overpriced (see Scheinkman and Xiong (2003), Nagel (2005)). Even though the evidence from empirical literature is mixed and somewhat weak, more recent findings claim that Miller’s hypothesis is confirmed by the data (see Boehme et al. (2005), also an extensive review of previous work can be found there). In my paper, I provide another story of overvaluation that is based on market selection. I show that if beliefs which earn a higher return get higher market shares, then under some general conditions the asset will become overvalued.

The paper is structured in the following way: Section 2 describes the model. Section 3 solves the “deterministic skeleton” of the model, that is in the case when the evolution depends on the conditional expectation of individual profits. Section 4 shows that the results obtained in Section 3 are robust to small perturbations. The most technical proofs are contained in the Appendix.

2 Model

2.1 Basic Setup

The framework that I base my model on is quite standard in the literature. I consider an economy with one riskless and one risky asset. The riskless asset is available in a perfectly
elastic supply at a price of 1, and its return is equal to $R > 1$ every period. The risky asset has price $p_t$ and pays dividends $y_t = y + \varepsilon_t$. I assume that $y$ is a constant and $\varepsilon_t$ are i.i.d. with zero mean and finite variance. The supply of the risky asset is equal to $S > 0$.

There is a continuum of agents in the economy and all agents live for two periods. They are born with an initial endowment $W^0$ of the riskless asset, and they consume only when they are old. To transfer wealth between periods agents use the stock market. Thus every period, trade happens between old and young generations. The old generation sells its portfolio to use the profit for consumption, whereas the young generation forms its portfolio to transfer wealth into the next period. We normalize the mass of one generation to be equal to one. Thus each period there is a continuum of mass one of young agents and a continuum of mass one of old agents.

Young agents are different in their beliefs about the distribution of the next period price and dividends of the risky asset. I assume that young agents have mean-variance utility functions and so their beliefs can be described in terms of means and variances of returns. Following Brock and Hommes, beliefs about the variance of returns, $\sigma^2$, are assumed to be the same across all agents and across all periods. As for beliefs about the means, I assume that they belong to a fixed finite set $\{\rho_1, \ldots, \rho_H\}$. Without loss of generality beliefs are ordered with respect to their optimism so that $\rho_1 < \cdots < \rho_H$.

I will refer to a group of agents with the same beliefs as a type. In total there are $H$ possible types: $(\rho_1, \sigma^2), \ldots, (\rho_H, \sigma^2)$ and a representative type is denoted as $h$. In period $t$ a young agent of type $h$ believes that in period $t + 1$ the distribution of price and dividends of the risky asset, $p_{t+1} + y_{t+1}$, has mean $\rho_h$ and variance $\sigma^2$.

Except for the difference in beliefs, all young agents have the same mean-variance utility function

$$U_h(W_{h,t+1}) = E_h(W_{h,t+1}) - (a/2)V(W_{h,t+1}).$$

Here, $W_{h,t+1}$ is the wealth that a young agent of type $h$ will earn in period $t + 1$, $E_h(W_{h,t+1})$ and $V(W_{h,t+1})$ are mean and variance of $W_{h,t+1}$ given beliefs of type $h$, and $a$ is a risk-aversion coefficient which we will normalize to 1. Since the variance is the same across agents and time periods, it is not indexed.

If a young agent of type $h$ in period $t$ buys $z_{h,t}$ units of the risky asset then his wealth in the next period, when he is old, is equal to

$$W_{h,t+1} = R(W^0 - p_t z_{h,t}) + (p_{t+1} + y_{t+1}) z_{h,t} = RW^0 + (p_{t+1} + y_{t+1} - Rp_t) z_{h,t}. \quad (1)$$

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1 Overlapping generation framework in the environment with heterogenous beliefs has been also used in Böhm and Wenzelburger (2005) and Böhm and Chiarella (2005).
Denote \( p_{t+1} + y_{t+1} - Rp_{t} \) as \( R_{t+1} \) so that \( W_{h,t+1} = RW^{0} + R_{t+1}z_{h,t} \). Then \( E_{h}(W_{h,t+1}) = RW^{0} + E_{h}(R_{t+1})z_{h,t} = RW^{0} + (\rho_{h} - Rp_{t})z_{h,t} \) and \( V(W_{h,t+1}) = z_{h,t}^{2} \cdot V(R_{t+1}) = z_{h,t}^{2}\sigma^{2} \). The maximization problem for type \( h \) becomes

\[
\max_{z} RW^{0} + E_{h}(R_{t+1})z - \frac{1}{2}\sigma^{2}z^{2},
\]

and so the demand function of type \( h \) is

\[
z_{h}(p_{t}) = \frac{1}{\sigma^{2}}E_{h}(R_{t+1}) = \frac{1}{\sigma^{2}}(\rho_{h} - Rp_{t}).
\]

Denote a mass of young individuals of type \( h \) at period \( t \) as \( n_{h,t} \). Since the mass of the young generation is normalized to 1 it is always the case that \( \sum_{h} n_{h,t} = 1 \). Thus \( n_{t} = (n_{1,t}, \ldots, n_{H,t}) \in \Delta^{H-1} \), where \( \Delta^{H-1} \) is an \((H - 1)\)-dimensional unit simplex, and it represents the distribution of beliefs in the market in period \( t \). In equilibrium the market for the risky asset must clear so

\[
S = \sum_{h} z_{h}(p_{t}) \cdot n_{h,t} = \frac{1}{\sigma^{2}} \left( \sum_{h} n_{h,t}\rho_{h} - Rp_{t} \right).
\]

From (4), the market-clearing price for the risky asset in period \( t \) is equal to

\[
p_{t} = \frac{1}{R} \left( \sum_{h} n_{h,t}\rho_{h} - \sigma^{2}S \right).
\]

From equation (5) it follows that the market-clearing price is determined by the distribution of beliefs \( n_{t} \) in the market. As the distribution of beliefs varies from the most pessimistic \((n_{1,t} = 1)\) to the most optimistic one \((n_{H,t} = 1)\) the market-clearing price, \( p_{t} \), will vary between \((\rho_{1} - \sigma^{2}S)/R \) and \((\rho_{H} - \sigma^{2}S)/R \). To ensure that for any distribution of beliefs the market-clearing price is positive I will assume:

**Assumption 2.1.** \((\rho_{1} - \sigma^{2}S)/R > 0 \).

To relate this setting to the rational-expectations model, notice that under rational expectations all agents in period \( t \) must have correct and, therefore, identical beliefs so that \( E_{1}(p_{t+1} + y_{t+1}) = \cdots = E_{H}(p_{t+1} + y_{t+1}) = p_{t+1} + y \). Plugging it into the pricing equation (5) we have

\[
Rp_{t} = p_{t+1} + y - \sigma^{2}S.
\]

Using the market equilibrium equation repeatedly and assuming that the standard transversality condition holds, the price of the risky asset is uniquely determined by

\[
p_{t} = \sum_{k=1}^{\infty} \frac{y - \sigma^{2}S}{R^{k}} = \frac{y - \sigma^{2}S}{R - 1}.
\]
We will denote the price in the model with rational expectations as \( p_r \), that is
\[
p_r = \frac{y - \sigma^2 S}{R - 1}.
\]

If agents were risk-neutral then the market price would be \( p_n = y/(R - 1) \). Not surprisingly \( p_n \) is higher than \( p_r \) because risk-averse agents demand a risk premium for holding the risky asset.

In the next section I introduce an evolutionary dynamic into the model. If the dynamic is to bring the economy to the equilibrium with rational expectations the asset price should converge to \( p_r \). The main result of Section 3, however, is that whenever \( p_n \) can be a market-clearing price\(^2\) the asset price will converge to the risk-neutral fundamental price, \( p_n \), even though agents themselves are not risk-neutral. In this sense the asset will become overvalued.

### 2.2 Evolutionary dynamics

Assumption 2.2 below introduces the evolution function \( f \) that governs how the distribution of beliefs evolves with time and it specifies the properties of the evolutionary dynamic.

I will denote the distribution of beliefs of young agents \((n_{1,t}, \ldots, n_{H,t})\) as \( n_t \), and the wealth vector \((W_{1,t}, \ldots, W_{H,t})\) as \( W_t \) so that \( W_t \) is the vector of wealth earned by the old generation in period \( t \). Finally, in what follows I will interchangeably use terms shares, weights and market shares when referring to \( n_t \) and its components.

**Assumption 2.2.** The evolutionary dynamic satisfies the following properties:

\( \text{(A0)} \) The share vector in period \( t, n_t \), is determined by a deterministic function \( f \) of wealth vector, \( W_t \), shares vector, \( n_{t-1} \), and some other variables, \( \xi_{t-1} \in \Xi \), where \( \Xi \) is a compact set. That is
\[
n_t = f(n_{t-1}, W_t, \xi_{t-1}),
\]
where \( f : \Delta^{H-1} \times \mathbb{R}^H \times \Xi \to \Delta^{H-1} \).

\( \text{(A1)} \) (Continuity) Function \( f \) is continuous with respect to all of its arguments and \( f_h \) — which is the \( h \)\textsuperscript{th} component of \( f \) — is Lipschitz continuous with respect to \( W_{h,t} \) so that
\[
|f_h(W_{h,t}, \ldots) - f_h(W_{h,t}', \ldots)| \leq K|W_{h,t} - W_{h,t}'| \quad \text{for any } h.
\]

\( \text{(A2)} \) (Weak monotonicity) For given \( W_t \) and \( n_{t-1} \) if \( n_{h,t} = f_h(n_{t-1}, W_t, \xi_{t-1}) > n_{h,t-1} \) for some type \( h \), then \( n_{h',t} \geq n_{h',t-1} \) for any type \( h' \) such that \( W_{h',t} \geq W_{h,t} \). Furthermore, if \( W_{h',t} \geq W_{h,t} \) and \( n_{h,t-1} > 0 \) then \( n_{h',t} > n_{h',t-1} \).

\(^2\)That is whenever \( p_n \in [1/R(\rho_1 - \sigma^2 S), 1/R(\rho_H - \sigma^2 S)] \) so that there exists a distribution of beliefs for which \( p_n \) clears the market.
(A3) (Non-triviality) If the vector of shares does not change then either all types with positive shares earned the same profit, or one type has a share equal to one.

(A4) (Evolution speed) The evolution process is sufficiently slow so that

\[ K < \frac{\sigma^2}{H \rho_H^2}. \]

Axiom (A0) specifies the variables that determine the evolutionary dynamic. It says that a new vector of shares, \( n_t \), is determined by the wealth vector, \( W_t \), shares in the previous period, \( n_{t-1} \), and possibly by some other parameters, \( \xi_{t-1} \), that are known by the end of period \( t - 1 \). For example, \( \xi_{t-1} \) could include wealth and share vectors from earlier periods as long as \( \Xi \) is a compact set.

The continuity axiom is a technical assumption. It states that \( f \) should be a continuous function and \( f_h \) should be Lipschitz continuous with respect to \( W_{h,t} \).

The weak monotonicity axiom is the assumption that implements the Friedman selection mechanism. Its requirements are quite weak. What it says is that if type \( h \) has strictly increased its share then all types that were more successful and earned higher wealth will also strictly increase their shares. The only exception is if \( W_{h',t} \geq W_{h,t} \) and \( n_{h',t-1} = 0 \). In this case it is allowed that \( n_{h',t} = n_{h',t-1} = 0 \). In other words, if a particular type is dead it is allowed (but not required) for this type to remain dead.

The non-triviality axiom is needed to remove trivial evolutionary rules that do not change the type shares even though the types earned different profit. Obviously, if the evolution does not change the shares of the types, then the market clearing price can potentially stay constant at any arbitrary level, and results in Section 3 would be invalid.

The last axiom ensures that the evolution speed is slow. It requires that other things being equal as we change the wealth of type \( h \), its share does not change too fast. The requirement of low evolution speed is quite typical. The reason is that otherwise the evolutionary dynamic becomes less robust since many types can very quickly die out because of few big shocks.

Example 2.3. An explicit example of a dynamic satisfying Assumption (2) is

\[ n_{h,t} = n_{h,t-1} + \zeta \cdot (W_{h,t} - \overline{W}_t)n_{h,t-1}, \]

where \( \overline{W}_t = \sum_h n_{h,t-1}W_{h,t} \). An additional requirement, is that \( \zeta \) should be sufficiently small in order to satisfy (A4) and to guarantee that \( \{n_{h,t}\} \) are between 0 and 1.

\[ ^3 \text{An alternative specification would be to assume that } f \text{ depends on } W_{t-1} \text{ instead of } W_t. \text{ The main results of the paper remain unchanged, however, proofs become considerably more complicated since the dynamic will possess an additional state variable. Proofs are available upon request.} \]
In the literature there have been considered many different dynamics that would satisfy axioms (A0)-(A4), including a discrete-time version of a replicator dynamic in Weibull (1995) (see also DSSW (1990), Sethi and Franke (1995), Branch and McGough (2005) and others). On the other hand, some dynamics, including the probabilistic choice rule would not satisfy Assumption 2. The easiest way to see the difference is to notice that in my paper if all types make the same profit, then their shares do not change. According to the probabilistic choice rule, if all types make the same profit their shares become equal.

Definition 2.4. Equilibrium is a sequence \( \{(n_t, p_t)\}_{t=1}^{\infty} \) such that for each period \( t \)

i) the market for the risky asset clears given \( p_t \) and \( n_t \);

ii) the distribution of beliefs evolves according to the evolution function \( f \), that is \( n_t = f(n_{t-1}, W_t, \xi_{t-1}) \).

System (8) recursively defines the equilibrium dynamic:

\[
\begin{align*}
    p_t &= \frac{1}{R} \left( \sum_h n_{h,t} \rho_h - \sigma^2 S \right) \\
    n_t &= f(n_{t-1}, W_t, \xi_{t-1})
\end{align*}
\]

(8)

The first equation is the market-clearing condition (5) and it ensures that market for the risky asset clears each period. The second condition requires that the distribution of beliefs evolves according to \( f \). In the second equation wealth vector \( W_t \) is determined by \( R_t \) which in turn depends on \( p_t \) and \( \varepsilon_t \). In particular, this means that \( p_t \) and \( n_t \) enter the system both on the left- and the right-hand sides and so (8) defines \( (p_t, n_t) \) only implicitly.

The dynamic unfolds in the following manner. At period one there is an initial distribution of beliefs \( n_1 \). Given \( n_1 \) the market-clearing price \( p_1 \) is determined by the first equation of (8). Given \( p_1 \) and given their beliefs young agents form portfolios according to (3). In period 2, system (8) will simultaneously determine \( n_2 \) and \( p_2 \) given \( (n_1, p_1) \) and \( \varepsilon_2 \). If given \( (n_1, p_1) \) a solution to (8) exists then in period 2 markets will clear and the distribution of beliefs will evolve according to \( f \). Similarly, having \( (n_2, p_2) \) we can solve (8) to determine \( (n_3, p_3) \) and so on.

Example 2.5. If the evolution function \( f \) has a functional form (7) then it is very easy to re-write (8) so that it defines the equilibrium dynamic explicitly. For the sake of example assume that \( \sigma^2 = 1 \) in which case (8) becomes

\[
\begin{align*}
    p_t &= \frac{1}{R} \left( \sum_h n_{h,t} \rho_h - S \right) \\
    n_{h,t} &= n_{h,t-1} + \zeta \cdot (W_{h,t} - \bar{W}_t) n_{h,t-1}.
\end{align*}
\]

(9)

In most of the examples above, one needs to impose additional assumptions on parameters usually in order to satisfy (A4). This is similar to the requirement of \( \zeta \) being sufficiently small for (7).
Using the definition of \( W_{h,t} \) and \( \bar{W}_t \) we can re-write the second equation as \( n_{h,t} = n_{h,t-1} + \zeta R_t (z_{h,t-1} - S) n_{h,t-1} \). By subtracting the first equation of \( 9 \) for period \( t-1 \) from the first equation of \( 9 \) for period \( t \) we have

\[
R(p_t - p_{t-1}) = \sum_h (n_{h,t} - n_{h,t-1}) \rho_h = \zeta R_t \sum_h (z_{h,t-1} - S) n_{h,t-1} \rho_h = \\
= \zeta R_t \sum_h (\rho_h - R p_{t-1} - S) n_{h,t-1} \rho_h = \\
= \zeta R_t \left( \sum_h \rho_h^2 n_{h,t-1} - \left( \sum_h \rho_h n_{h,t-1} \right)^2 \right) = \zeta R_t A_{t-1}.
\]

Here to get from the second line to the third we used that \( R p_{t-1} = \sum_h n_{h,t-1} \rho_h - S \). Term \( A_{t-1} \) denotes the expression in big parenthesis. Using that \( R_t = p_t + y_t - R p_{t-1} \) we can solve for \( p_t \):

\[
p_t = \frac{1 - \zeta A_{t-1}}{R - \zeta A_{t-1}} R p_{t-1} + \frac{\zeta A_{t-1}}{R - A_{t-1}} y_t.
\]

Thus given variables in period \( t-1 \) and the dividend shock \( \varepsilon_t \) we can use \( 10 \) to determine \( p_t \) and then using the evolution function \( 7 \) we can determine \( n_t \) and so on.

While given this particular functional form of \( f \) it is possible to solve for \( (n_t, p_t) \) directly, in general, system \( 8 \) determines \( (n_t, p_t) \) only implicitly. To ensure that an equilibrium dynamic is well-defined we need to verify that for any share vector \( n_{t-1} \) and price \( p_{t-1} \) the solution to \( 8 \) exists.

**Statement 2.6.** For any \( (n_{t-1}, p_{t-1}) \) such that \( n_{t-1} \in \Delta^{H-1} \) and \( p_{t-1} > 0 \) there exists \( (n_t, p_t) \) that is a solution to \( 8 \) and such that \( n_t \in \Delta^{H-1} \) and \( p_t > 0 \).

\( \triangleright \) The proof of this statement is a simple continuity argument. The market clearing price \( p_t \) is determined from the equation \( \sum_h n_{h,t}(p) \cdot z_h(p) = S \), where the LHS is total demand, \( TD(p) \). On one hand

\[
TD(0) = \sum_h n_{h,t}(0) \cdot z_h(0) \geq z_1(0) > S,
\]

where the first inequality follows from the fact that \( \rho_1 < \cdots < \rho_H \) and the second one follows from Assumption \( 21 \). On the other hand when \( p \) is large (greater than \( \rho_H / R \)), the demands of all types will be negative, and thus the total demand will be less than \( S \). Given (A1), the total demand is a continuous function of \( p \), and thus there exists a market-clearing price \( p_t \) and \( p_t > 0 \). The fact that \( n_{h,t} \in \Delta^{H-1} \) follows from (A0). \( \triangleright \)

The Statement above guarantees that the solution to \( 8 \) exists. I do not require the uniqueness as all theorems and statements in the paper hold for any sequence \( \{ (n_t, p_t) \}_{t=1}^{\infty} \) that is generated by \( 8 \), regardless of whether \( 8 \) has a unique solution for each \( t \) or not.
I conclude the section by highlighting the role of the assumption that restricts the set of possible beliefs to set \{\((\rho_1, \sigma_1), \ldots, (\rho_H, \sigma_H)\)\}. Given this assumption, the market outcome is fully determined by the distribution of agents’ beliefs over this set. The evolution function \(f\) governs the dynamic of the system and puts restrictions on how the belief distribution changes with time. Importantly, the restrictions imposed by \(f\) are the only restrictions that are imposed on the dynamic. Thus, the evolution is determined on the aggregate level of beliefs’ distribution and not on the individual level. In particular, there are no explicit constraints on individual evolution of beliefs as long as the change in the aggregate distribution of beliefs is consistent with the axioms of Assumption 2.2.

A more standard approach adopted in the literature is to model the evolution at the level of individual learning/forecasting rules. For example, in Hommes et al. (2005) and Chiarella and He (2001) there are two types of agents. One type forms its expectations based on the extrapolation of the trend, and the other type believes that the price will go back to fundamentals. While the choice of these particular rules is supported by psychological evidence there are two reasons why I decided not to use this approach. First, there are many different classes of rules that could be supported by psychological findings or by economic intuition. However, a priori, it is not clear what set of rules would be the best to model the stock market, and how robust the results are to different choices of rules. The second reason is that the evolution of three rules is already difficult to analyze analytically. Considering more than three rules becomes almost impossible. With my approach, I avoid these problems. By modeling the evolution of beliefs not on the individual but on the aggregate level, I can allow for a more general class of dynamics and with any number of types.

3 Deterministic Evolution

The evolution process defined in the previous section depends on the realization of random dividend payoffs, and so the whole system evolves according to some stochastic process. In this section, as it is often done in the literature (see e.g. Brock and Hommes (1998), Brock et al. (2005)), I will analyze the “deterministic skeleton” of this process which is the dynamic that arises from (8) when \(\varepsilon_t = 0\) for all \(t\) so that \(y_t = y\). In this case \(R_t\) becomes \(p_t + y - Rp_{t-1}\) and \(W_{h,t}\) becomes \((p_t + y - Rp_{t-1})z_{h,t-1} + W^0R\). To separate the deterministic case considered in this section and a stochastic dynamic considered in the next section I will denote \(p_t + y - Rp_{t-1}\) as \(\tilde{R}_t\) and \((p_t + y - Rp_{t-1})z_{h,t-1} + W^0R\) as \(\tilde{W}_{h,t}\). An economic interpretation of a deterministic skeleton dynamic is that the evolution process depends on the conditional expectation of wealth and not on the actual wealth realization.

The first property of the equilibrium dynamic that I derive is that if all types make the same profit then their shares do not change. The proof is a simple implication of the weak
monotonicity assumption.

**Statement 3.1.** If all agents make the same profit in period $t$ then $n_{h,t} = n_{h,t-1}$ for any $h$. In particular, this holds if $\tilde{R}_t = 0$.

Assume not. If the shares of some types have changed then there is type $h$ that has increased its share, that is $n_{h,t} > n_{h,t-1}$. By weak monotonicity $n_{h',t} \geq n_{h',t-1}$ for any other type $h'$. But then $\sum_h n_{h,t} > \sum_h n_{h,t-1} = 1$ which is impossible. If $\tilde{R}_t = 0$ then the wealth of each type, $\tilde{W}_{h,t}$, is equal to $W^0 R$, and thus all types have the same wealth.

As we will show, the properties of the equilibrium dynamic depends on the support of agents’ beliefs, that is, on $\rho_1, \ldots, \rho_H$. We distinguish the following three cases:

**Definition 3.2.** We say that beliefs in the economy are **diverse** if there exists a distribution of beliefs such that $p_n = y / (R - 1)$ is a market-clearing price. That is if

$$\frac{1}{R} (\rho_1 - \sigma_2^2 S) \leq \frac{y}{R - 1} \leq \frac{1}{R} (\rho_H - \sigma_2^2 S). \quad (11)$$

Beliefs are said to be **too low** if the highest possible market-clearing price is smaller than $p_n$, that is $1/R \cdot (\rho_H - \sigma_2^2 S) < p_n$. Beliefs are said to be **too high** if the lowest possible market-clearing price is greater than $p_n$, that is $1/R \cdot (\rho_1 - \sigma_2^2 S) > p_n$.

The last two cases in the definition correspond to the situation when everyone in the society is either too pessimistic or too optimistic. Then the risk-neutral fundamental price, $p_n$, can never be a market-clearing price for any distribution of beliefs over $\rho_1, \ldots, \rho_H$. In the case of diverse beliefs the risk-neutral price is attainable, that is, there is a distribution of beliefs such that the risk-neutral price clears the market. While formally I consider all three cases, the last two are unrealistically restrictive and most importantly are not robust to the introduction of new beliefs.

As the next theorem shows, whenever beliefs in the society are diverse — that is $p_n$ can potentially be a market-clearing price — the market will converge to it. Furthermore, if inequalities in (11) are strict then the heterogeneity of beliefs will persist even in the long-run. The important implication of these results is that the risky asset becomes overvalued as compared to its price in the equilibrium with rational expectations. When beliefs are either too low or too high the heterogeneity will not persist and only one type will survive. If beliefs are too high it will be the type with the lowest belief, and if beliefs are too low it will be the type with the highest belief.

The intuition behind this result is based on the DSSW insight and to see it consider the following stylized example. Assume that beliefs in the society are diverse and there are several types with positive shares. Assume also that $p_t = p_{t+1} = p_e$, that is the market-clearing prices today and tomorrow are equal to the equilibrium price with rational expectations.
I argue that this price sequence cannot belong to the equilibrium dynamic. Indeed, given $p_t = p_{t+1} = p_r$, the risky asset has positive expected excess return, $\tilde{R}_{t+1} > 0$. Agents who belong to more optimistic types will have more units of the asset in their portfolios (see (3)), and in expectation will earn more money. Compared to period $t$ their market share will grow and so will the asset price $p_{t+1}$. Thus along the equilibrium path it should be the case that $p_{t+1} > p_t = p_r$. Because the evolution is slow, $p_{t+1}$ should be close to $p_t$ and so, in fact, we can apply the same logic to show that $p_{t+2} > p_{t+1}$ and $\tilde{R}_{t+2} > 0$. Optimists’ shares will grow again as well as the market-clearing price $p_{t+2}$. As the asset price approaches the risk-neutral price the asset excess return disappears and so does the advantage of optimists. In the long-run the market-clearing price converges to $p_n$.

Importantly, in the stylized example above if a particular type of agents would have correct beliefs they would not be willing to hold an optimistic portfolio. The reason is that optimistic portfolios despite being more profitable involve more risk than agents with correct beliefs would be willing to take. Consequently, agents with correct beliefs as well as agents with pessimistic beliefs earn less money and, to some extent, lose the market influence.

**Theorem 3.3.** Assume that shares of all types are positive in $t = 1$.

i) If beliefs are diverse then for any equilibrium sequence $\{(n_t, p_t)\}_{t=1}^{\infty}$ the market-clearing price $p_t$ will converge to $p_n$. Furthermore, if inequalities in (11) are strict then at least two types will co-exist in the long-run.

ii) If beliefs are too high (too low) then only the highest (lowest) type will survive in the long-run. The long-run price will be determined by beliefs of the survived type.

The role of the assumption that $n_{1,h} > 0$ for each $h$ is to rule out the following case. Assume that beliefs are too low and consider an equilibrium dynamic given these beliefs and some initial conditions. From Theorem 3.3 it follows that only type $H$ will survive. Now add a very high belief $\rho'$ with share that is always equal to zero. Clearly the equilibrium dynamic will not change even though formally beliefs will become diverse. The reason is that even though the added type $\rho'$ makes beliefs diverse, it is actually redundant and has zero effect on the price. Assuming that shares of all types are positive in the initial period allows me to rule out this pathological case.\textsuperscript{5,6}

\textsuperscript{5}The assumption that shares of all types are positive in $t = 1$ is, in fact, somewhat stronger than needed. An alternative way to remove this pathological case would be to use the following definition. We say that beliefs are diverse along the path $\{(p_t, n_t)\}$ in period $t$ when the fundamental price $p_n$ could be a market clearing price given the beliefs of only those types that have positive shares in period $t$. In other words, if there is $n_t'$ such that $n_{h,t}' > 0$ iff $n_{h,t} > 0$ and $p_n = \frac{1}{\pi}(\sum_h n_{h,t}'\rho_h - \sigma^2 S)$. Then Theorem 3.3 can be re-formulated as: “If there is such $t$ so that beliefs are diverse along the path $\{(p_t, n_t)\}$ in period $t$ then $p_t$ will converge to $p_n$.” The reason why I do not use this formulation is that in addition to being more complicated it is stated in terms of endogenous properties of the equilibrium path.

\textsuperscript{6}Even with the assumption that all types have positive shares in the initial period it is still possible to
In the proof of theorem 3.3, I will use three lemmas. Only the proof of the first lemma is provided in the main text as it is very intuitive and demonstrates the role of the weak monotonicity axiom. The last two lemmas are somewhat technical and will be proved in the appendix.

**Lemma 3.4.** In equilibrium \( \tilde{R}_t \) and \( \Delta p_t = p_t - p_{t-1} \) have the same sign, unless there is a type \( h \) so that \( n_{h,t} = n_{h,t-1} = 1 \). In the latter case \( \Delta p_t = 0 \) regardless of \( \tilde{R}_t \).

Intuitively, when \( \tilde{R}_t \) is positive the optimists earn higher profit. From the weak monotonicity axiom when optimists earn higher profit their shares increase and so the market-clearing price should increase as well.

**Proof of Lemma 3.4.** The second part of the Lemma is trivial: if \( n_{h,t} = n_{h,t-1} = 1 \) for some \( h \), then from (5) it follows that \( p_t = p_{t-1} \) regardless of \( \tilde{R}_t \). Now consider the case there is no such type \( h \).

Since types are ordered with respect to their optimism \( z_1(p) < z_2(p) < \cdots < z_H(p) \) for any \( p \). Given that, from \( \tilde{W}_{h,t} = \tilde{R}_t z_{h,t-1} + W^0 R \) follows that if \( \tilde{R}_t > 0 \) then \( \tilde{W}_{1,t} < \tilde{W}_{2,t} < \cdots < \tilde{W}_{H,t} \), and if \( \tilde{R}_t < 0 \) then \( \tilde{W}_{1,t} > \tilde{W}_{2,t} > \cdots > \tilde{W}_{H,t} \). By taking the market-clearing conditions for periods \( t \) and \( t-1 \) and subtracting one from the other we get the following equation

\[
R \Delta p_t = \sum_h \rho_h (n_{h,t} - n_{h,t-1}). \tag{12}
\]

From Statement 3.1 we know that if \( \tilde{R}_t = 0 \) then the RHS of (12) is equal to zero, and thus \( p_t = p_{t-1} \). This proves that \( \tilde{R}_t = 0 \Rightarrow \Delta p_t = 0 \).

Consider the case when \( \tilde{R}_t > 0 \). By weak monotonicity \( n_{1,t} \leq n_{1,t-1} \). Otherwise if \( n_{1,t} > n_{1,t-1} \), all types would weakly increase their shares with at least one (for \( h = 1 \)) change being strict. By non-triviality \( n_t \neq n_{t-1} \). Indeed, when \( \tilde{R}_t > 0 \) all types earn different wealth and we are considering the case when there is no type \( h \) such that \( n_{h,t} = n_{h,t-1} = 1 \). Thus, there is at least one type that will strictly increase its share between \( t-1 \) and \( t \) and it cannot be type 1.

Let \( h' > 1 \) be the smallest index among types that strictly increased their shares. By weak monotonicity \( n_{h,t} \geq n_{h,t-1} \) for all \( h > h' \), and, by definition of \( h' \), \( n_{h,t} \leq n_{h,t-1} \) for all model appearance of new beliefs. For example, assume we want to model the situation where agents can have a belief \( \rho' \) but only when \( t > T \). To do that we add \( \rho' \) to the original set \( \{\rho_1, \ldots, \rho_H\} \), and keep a share of agents with these beliefs, \( n_i' \), less than \( \varepsilon \) as long as \( t \leq T \). By choosing \( \varepsilon \) an arbitrarily small we can make the price impact of agents with \( \rho' \) beliefs negligible. Note that Assumption 2.2 allows us to ensure that we can keep \( n_i' \) as small as we need. The only axiom that forces us to increase \( n_i' \) is (A2). However, it does not specify how much \( n_i' \) should increase. (A2) only requires that under some circumstance it should increase. In particular, we could set \( n_i' = \varepsilon/2 \) and then, when necessary, increase it respectively by \( \varepsilon/4, \varepsilon/8 \), and so on until moment \( T \) comes.
\( h < h' \). Then, from (12)

\[
R \Delta p_t = \sum_{h=1}^{H} (n_{h,t} - n_{h,t-1}) \rho_h = \sum_{h=1}^{h'-1} (n_{h,t} - n_{h,t-1}) \rho_h + \sum_{h=h'}^{H} (n_{h,t} - n_{h,t-1}) \rho_h > \\
> \rho_{h'} \sum_{h=1}^{h'-1} (n_{h,t} - n_{h,t-1}) + \rho_{h'} \sum_{h=h'}^{H} (n_{h,t} - n_{h,t-1}) = 0.
\]

The inequality is strict because there are at least two types that strictly change shares. Thus \( \tilde{R}_t > 0 \Rightarrow \Delta p_t > 0 \). The case of \( \tilde{R}_t < 0 \) is identical.

Lemma 3.5. Price sequence converges.

Proof of Lemma 3.5 The proof of the lemma is given in the appendix.

Given that the price sequence converges we can conclude that \( \tilde{R}_t = p_t + y - Rp_{t-1} \) also converges. Denote the limit of \( \tilde{R}_t \) as \( \tilde{R}_\infty \). The next lemma shows that if \( \tilde{R}_\infty \neq 0 \) then only one type survives in the end. The proof is somewhat technical but the idea is simple. If, say \( \tilde{R}_\infty > 0 \), then \( \tilde{R}_t \) eventually becomes positive. The most optimistic type then will be earning the highest wealth and by weak monotonicity its share will be increasing. Thus \( \{n_{H,t}\} \) converges and, furthermore, we can show that it converges to one.

Lemma 3.6. Take any equilibrium path \( \{(n_t, p_t)\} \) and consider sequence \( \{\tilde{R}_t\} \) along this path:

i) If there exists period \( T^0 \) and a constant \( c > 0 \) such that either \( \tilde{R}_t > c > 0 \) for any \( t \geq T^0 \) or \( \tilde{R}_t < -c < 0 \) for any \( t \geq T^0 \) then only one type survives in the limit.

ii) If \( \tilde{R}_t \to \tilde{R}_\infty \) and \( \tilde{R}_\infty \neq 0 \) then only one type survives in the limit.

Proof of Lemma 3.6. The proof of the lemma is given in the appendix.

Having established these three Lemmas we can complete the proof of the theorem. First, consider the case when beliefs are too low. Then it has to be the case that \( p_\infty < p_n \) and so \( \tilde{R}_\infty > 0 \). From Lemma 3.6 it follows that only one type survives in the limit. In fact, we can show that it will be the most optimistic type. Indeed, from weak monotonicity it follows that if \( \tilde{R}_t > 0 \) then \( n_{H,t} \geq n_{H,t-1} \). When beliefs are too low, \( \tilde{R}_t > 0 \) for any \( t \) and so sequence \( \{n_{H,t}\} \) is weakly increasing and converges. Given Lemma 3.6 it can converge only to 1 or to 0. But the latter is impossible given the assumption \( n_{H,1} > 0 \). The case of high beliefs is similar. This proves part ii) of the theorem.

Now consider the case of diverse beliefs. I argue that \( \tilde{R}_\infty = 0 \) which means that \( 0 = \tilde{R}_\infty = p_\infty + y - Rp_\infty \) and so \( p_\infty = p_n \). Assume not. Consider the case of \( \tilde{R}_\infty > 0 \) (case of \( \tilde{R}_\infty < 0 \) is similar). By Lemma 3.6 only one type survives in the limit and using the same logic as in the paragraph above we can show that it will be type \( H \). When \( n_{H,t} \) converges to 1 the
market price $p_t$ will converge to $(\rho_H - \sigma^2S)/R$. Since beliefs are diverse $(\rho_H - \sigma^2S)/R \geq p_n$. If $(\rho_H - \sigma^2S)/R = p_n$ then $p_t \rightarrow p_n$ and so $\tilde{R}_t \rightarrow 0$, which is a contradiction since we consider the case $\tilde{R}_\infty > 0$. If $(\rho_H - \sigma^2S)/R > p_n$ there exists $T$ such that $p_t > p_n$ for any $t > T$ and $|p_t - p_{t-1}| < \varepsilon$ where $\varepsilon$ is chosen so small that $\tilde{R}_t < 0$. Such $\varepsilon$ exists because for sufficiently large $t$ both $p_t$ and $p_{t-1}$ will be above $p_n$ and $\tilde{R}_t = p_t + y - Rp_t < 0$. But then as $n_{H,1}$ gets sufficiently close to 1, $\tilde{R}_t$ becomes negative, which is a contradiction since $\tilde{R}_\infty$ was assumed to be positive. Thus $\tilde{R}_\infty = 0$ and so $p_t \rightarrow p_n$.

Finally, we will show that heterogeneity will persist in the long-run when inequalities in (11) are strict. Let $(\rho_1 - \sigma^2S)/R < p_n < (\rho_H - \sigma^2S)/R$ and assume that only one type, type $h$, survives in the limit. Since $p_t \rightarrow p_n$ it has to be the case that $1 < h < H$. From Step 4 of Lemma 3.15 we know that the price sequence is monotone. Assume it is weakly increasing (a weakly decreasing case is similar). Then from Lemma 3.4 we know that $\tilde{R}_t \geq 0$ and so by weak monotonicity sequence $n_{H,1}$ is weakly increasing. Given that $n_{H,1} > 0$ it cannot converge to 0. This completes the proof of the theorem.

Theorem 3.3 shows that if the evolution process (or as it can also be called, the market selection mechanism) satisfies properties of Assumption 2.2 and beliefs are diverse then the market price of the risky asset will converge to the risk-neutral fundamental price. This is despite the fact that the agents themselves are risk-averse. The main driving force behind this result is the observation that optimists earn higher profit at the expense of taking higher risk. Since the evolution does not punish agents for taking excessive risk, optimists grow and bring the price up to the risk-neutral level. This reasoning suggests that what drives the result is the assumption that the evolution depends on actual wealth and not on risk-adjusted wealth. The natural question then is how appropriate is this assumption.

One can think of two explanations of the market selection mechanism. One explanation is that the wealth of successful types is higher, and thus their market weights should be higher. In this case, using the raw wealth for $f$ would be an appropriate assumption to make. Another explanation is that agents adopt the beliefs of more successful types. In this case, it is possible that agents are taking into account the ex-ante risk that was involved in different portfolio choices. However, to what extent this is true and how it would affect the evolution mechanism is not clear. First of all, as DSSW (1990) argue on p. 724, many investors are likely to attribute the higher return of an investment strategy to market timing skill rather than to its greater risk. Second, empirical literature has somewhat mixed evidence regarding the extent to which ex-ante risk is a factor in the selection mechanism.

Some of the ways in which empirical evidence supports the use of actual wealth include

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Footnote 7: Given the symmetry of the model, another (perhaps less plausible) scenario is possible where the initial asset price is so high that $\tilde{R}_t < 0$ in which case it is pessimists who grow in numbers and bring the price down to $p_n$.  

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the following. It is documented by many researchers that there is a strong relationship between the inflow of new investments into a mutual fund and the fund’s past performance (see Ippolito (1992), Gruber (1996), Chevallier and Ellison (1997), Sirri and Tuffano (1998) and many others). The common way to measure funds’ performance is to use risk-adjusted returns. However, when the raw returns are used to estimate the relationship, the results are highly similar. It is not surprising since raw and risk-adjusted returns are correlated and so one would serve as a proxy for another. Nonetheless, as some researchers independently noticed (Gruber (1996) and Sirri and Tuffano (1998)), the raw performance had an impact on fund flows which was separate from risk-adjusted measures. Moreover, if the risk measure is not correlated with the raw returns then its explanatory power can be very low. For example, in Sirri and Tuffano (1998) the standard deviation of past returns is used as an alternative risk measure. Despite the fact that it was the only risk measure publicly reported to the agents in the sample, it turned out to be only ”marginally significant” with $p$-value 0.105.

4 Robustness to Introduction of Random Shocks

In the previous section it was established that the deterministic skeleton of the evolutionary dynamic converges to the risk-neutral fundamental price, $p_n$, whenever it can be supported as a market-clearing price given beliefs $\{\rho_1, \ldots, \rho_H\}$. Consequently, the risky asset becomes overvalued as compared to the equilibrium with rational expectations. The main driving force for this result is the observation that if the asset is priced according to the rational expectation equilibrium then it pays positive excess return. Optimists who hold more units of the asset earn higher profit, increase their market weight and create an upward pressure on the asset price.

What optimists — or more precisely agents belonging to optimistic types — do that other agents cannot is they hold portfolios that are much riskier. They misperceive the tradeoff between risk and return and end up taking too much risk. The selection mechanisms considered in the last section do not punish such behavior and, in fact, some of them will actually encourage extremely optimistic types. For instance, according to evolution process in (7) types that earn larger profits gain more in their market shares. While being intuitive this property also implies that types with extremely optimistic beliefs will experience the largest growth.\footnote{In this discussion I implicitly consider a case of $\tilde{R}_t$ being positive so that it is optimists who have advantage. Due to the symmetry of the model the same effect is possible for pessimists. If the initial distribution of beliefs is such that the market-clearing price is very high so that $\tilde{R}_t < 0$ then it will be the type with extremely pessimistic beliefs that will experience the largest growth.} This point is illustrated on Figure 1 where the type with the most optimistic
yet the most “incorrect” belief ends up with the largest market share.

Figure 1: The most optimistic type ends up having the highest share. The horizontal line is the risk-neutral fundamental price 60. The risk-adjusted fundamental price is 40. Type beliefs are \((57, 60.5, 61.5, 62, 63)\). Other parameters used for simulation are \(R = 1.05, y = 3, S = 1, \sigma^2 = 1\). The evolution function is given by (7) with \(\zeta = 0.005\).

Figure 2: The most optimistic and pessimistic types die out. The horizontal line is the risk-neutral fundamental price 60. The risk-adjusted fundamental price is 40. The dividend shock \(\varepsilon_t \sim U[-1, 1]\). Type beliefs are \((57, 60.5, 61.5, 62, 63)\). Other parameters used for simulation are \(R = 1.05, y = 3, S = 1, \sigma^2 = 1\). The evolution function is given by (7) with \(\zeta = 0.005\).

As Figure 2 shows, this counterintuitive outcome disappears when we re-introduce shocks into the dynamic. It is based on the same selection mechanism as Figure 1, except that now the wealth argument of \(f\) is the realized wealth \(W_t\) and not \(\tilde{W}_t\). In this new setting the excessive risk-taking is being punished and types with the most extreme beliefs disappear. This observation is related to the Samuelson (1971) result which proved that the wealth generated by portfolios with higher risk and higher expected returns can be dominated with
probability one by wealth generated by less risky portfolios with lower returns.

Figure 2 demonstrates that Samuelson’s result is relevant to my model and the intuition is as follows. Agents with the most optimistic beliefs always have the largest demand for the risky asset and their portfolios therefore are always riskier than that of agents with moderate beliefs. Agents with extremely pessimistic beliefs also take too much risk because they choose to short the asset. By consistently holding the riskiest portfolios agents with extreme beliefs become susceptible to Samuelson’s result and can become extinct even if their expected returns are the highest.

One might think that the extinction of types with extreme beliefs can significantly undermine the optimists’ ability to drive prices up, since taking riskier positions can be punished by the market. However, the next theorem shows that the results from the previous sections are reasonably robust to the introduction of randomness into the evolutionary dynamic. More specifically, with probability one the equilibrium price will be absorbed into some interval around $p_n$, and the size of the interval is determined by the support of dividend shock $\varepsilon_t$. In particular, when it is small, that is dividend payout is close to its mean, the market-clearing price will be close to the risk-neutral price level.

**Theorem 4.1.** Assume that beliefs are diverse and that shares of all types are positive in $t = 1$. Assume also that $\varepsilon_t$ is distributed with support $[-M; M]$. Then with probability one there will be a moment $T \leq \infty$ such that for any $t \geq T$

$$p_n - \frac{M}{R - 1} \leq p_t \leq p_n + \frac{M}{R - 1},$$

$$\text{The proof is given in the Appendix.}$$

The requirement that $T \leq \infty$ and not just $T < \infty$ comes from the fact that there is a possibility that for some realization of $\{\varepsilon_t\}$ the price sequence will be a converging sequence with the limit being one of the boundaries.

The reason why Samuelson’s result does not substantially limit the optimists’ power to increase the price is that it was derived in the setting where the mean and the variance of returns are fixed. In my paper they are endogenously determined by prices. In particular, as the price goes up, the optimists demand less of the risky asset, while more cautious agents might think that the asset is too overvalued and short it. This means that the riskiness of the optimists’ portfolios decreases while the pessimists portfolios become more risky. However, it is still the case that optimists’ portfolios are more profitable. Thus optimists do not necessarily die out, and, furthermore, as Theorem 4.1 shows, for some parameter values they are still capable of pushing the price above $p_r$.

I conclude this section with a brief discussion of how the evolutionary dynamic generated by (8) is related to some stylized facts about financial markets. While designing the model
that would generate all particularities of financial data was not a primary goal of this paper, it is still interesting to see how the equilibrium dynamic presented in this paper will fit in the picture.

Since the model is very simplified, many stylized facts about consumption and dividend growth, as well as about real interest rates, cannot be represented in the model. Thus, I will concentrate on three stylized facts: equity premium, excess volatility and predictability. Intuitively, since the model in the paper predicts overvaluation the excess return should be low in the long-run. However, if most agents have relatively low beliefs in the beginning, then it is natural to expect that at first the risky asset will pay a high premium which will later decline. As for the last two properties, it is well-known that assuming learning in financial markets can lead to excess volatility and predictability (see Timmerman (1996)). Since the evolution of beliefs is similar to models with learning, we should expect both features to be present in my model.

As an illustration for the discussion above I will consider the following example. Let the evolution function be defined as

$$n_{h,t} = \frac{W_{h,t} n_{h,t-1}}{\sum_h W_{h,t} n_{h,t-1}} = \frac{R_t z_{h,t-1} + W^0 R}{R_t S + W^0 R}. \quad (14)$$

In words, the share of type $h$ in period $t$ is equal to the share of wealth of this type relative to the total wealth in the economy. When $W^0$ is sufficiently large (so that (A0) and (A4) are satisfied) this evolutionary dynamic satisfies Assumption 2.

Define the parameters of the model as follows. Set $R = 1.05, \sigma^2 = 1, S = 1, y = 3, W^0 = 120$ and assume that $\varepsilon_t \sim U[-0.13, 0.13]$. For these parameters $p_r = 40$ and $p_n = 60$. Set the support of beliefs to be equal to [45, 47, 48, 54, 64, 71], that is the number of types is six.

As the initial types distribution, we assume that each type has equal weight. I iterate the evolutionary dynamic for 1000 periods by which time the system appears to stabilize. As expected, the excess return declines with time. For the first 100 periods it is equal to 12% and the average for all 1000 periods is equal to 1%. The volatility of returns is 20.32% and it is much higher than the volatility of dividends which is approximately 7.67%. Finally, in the beginning, returns on the equilibrium path are predictable. In particular, when regressing $R_t$ on log($y_{t-1}/p_{t-1}$) the coefficient at log($y_{t-1}/p_{t-1}$) is positive and significant. However, as more observations are included the value of $R^2$ declines. For example, for the first 100 periods $R^2$ is 0.3088 for the first 200 periods it is 0.1553 and if the regression is taken over the whole sample then $R^2$ is just 0.02.

The parameter values used for this simulation were determined to match some of the real-data features (e.g. volatility of dividends being equal to 7%) and other than that they

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9In this paragraph, by $R^2$ I mean a standard econometric $R^2$ and not the square of the riskless return $R$ that is used in my model.
are not particularly special. As argued above observing excess volatility and predictability is something we should expect from this kind of model. The evolutionary nature of the dynamic provides a source of extra volatility and explains predictability, which is similar to the Timmerman (1996) result.

5 Conclusion

In the paper I provide a behavioral explanation of asset overvaluation. The behavioral part of the model is the assumption that agents do not have rational expectations. The model considered in the paper is symmetric in terms of agents’ beliefs and agents’ ability to affect prices. The latter means that there are no short-sale constraints and so bearish agents have an opportunity to decrease the asset price by shorting it. The overvaluation is achieved as a result of the market selection mechanism which is based on the idea proposed in Friedman (1953). The novelty of my approach is that I model the selection mechanism not at the level of individual learning rules but at the level of aggregated distribution of beliefs. The advantage of this approach is that it enables me to achieve results in a more general setting than is usually done in the literature. In particular, I analyze the situation with any finite number of types, and I do not place any explicit restrictions on individual learning rules.

The main result of the paper is that in an economy where agents have heterogeneous beliefs and there exists a market selection mechanism, the price of the risky asset converges to the risk-neutral price level, even though all agents are risk-averse. This means that the risky asset ends up being overpriced as compared to the equilibrium with rational expectations. The primary intuition is based on the DSSW (1990) insight, which is that even though optimists choose sub-optimal portfolios this sub-optimality means higher return at a price of higher risk. However, the selection mechanism depends on types’ wealth (in the paper it is either expected wealth or realized wealth), and so on average the number of optimists grow, which pushes the asset price towards the risk-neutral fundamental price. In the paper, I formalize this insight and show that it is valid for a large class of selection mechanisms.
6 Appendix. Proofs

Lemma 3.5 Price sequence converges.

In the proof of this Lemma I use $\tilde{W}_{h,t}$ to denote the period $t$ wealth vector of all types but type $h$. Similarly, I use $n_{-h,t-1}$ to denote a vector of shares of all types but type $h$ in period $t-1$.

Step 1: There exists $\tilde{W}_{h,t}'$ such that $n_{h,t-1} = f_h(\tilde{W}_{h,t}', n_{h,t-1}, \tilde{W}_{-h,t}, n_{-h,t-1}, \xi_{t-1})$. Moreover, if $\tilde{W}_{\min}$ and $\tilde{W}_{\max}$ denote the lowest and the highest wealth levels among $(\tilde{W}_{1,t}, \ldots, \tilde{W}_{H,t})$, then $\tilde{W}_{\min} \leq \tilde{W}_{h,t}' \leq \tilde{W}_{\max}$.

By weak monotonicity, $f_h(\tilde{W}_{\min}, n_{h,t-1}, \tilde{W}_{-h,t}, n_{-h,t-1}, \xi_{t-1}) \leq n_{h,t-1}$ since otherwise all types would weakly increase their shares with at least one increase (for type $h$) being strict. Similarly, $f_h(\tilde{W}_{\max}, n_{h,t-1}, \tilde{W}_{-h,t}, n_{-h,t-1}, \xi_{t-1}) \geq n_{h,t-1}$ since otherwise, there would be a type that would have strictly increased its share and that has wealth which is less than or equal to the wealth of type $h$. By continuity of $f_h$ there exists $\tilde{W}_{h,t}'$ such that $\tilde{W}_{\min} \leq \tilde{W}_{h,t}' \leq \tilde{W}_{\max}$ and that $n_{h,t-1} = f_h(\tilde{W}_{h,t}', n_{h,t-1}, \tilde{W}_{-h,t}, n_{-h,t-1}, \xi_{t-1})$.

Variable $\tilde{W}_{h,t}'$ defined in Step 1 is an auxiliary variable that will be used in Step 2. In addition to that we define two auxiliary variables $z_{h,t-1}'$ and $\rho_h'$ as follows. Given $\tilde{W}_{h,t}'$ let $z_{h,t-1}'$ be such that $\tilde{W}_{h,t}' = \tilde{R}_t z_{h,t-1}' + W^0 R$ and let $\rho_h'$ be such that $z_{h,t-1}' = \frac{1}{\sigma^2}(\rho_h' - Rp_{t-1})$. Formally speaking, $z_{h,t-1}'$ and $\rho_h'$ are just the solutions to the two equations above and, in particular, it is not assumed that $\rho_h' \in \{\rho_1, \ldots, \rho_H\}$. The economic meaning of $z_{h,t-1}'$ and $\rho_h'$ is as follows: given $p_{t-1}$ and $p_t$ an agent with belief $\rho_h'$ — regardless of whether or not $\rho_h' \in \{\rho_1, \ldots, \rho_H\}$ — would purchase $z_{h,t-1}'$ units of the risky asset and would earn $\tilde{W}_{h,t}'$ in period $t$.

Step 2: The goal of this step is to transform (12) into such a way that will later enable me to prove the convergence.

From the definition of $\tilde{W}_{h,t}'$, we have that $n_{h,t-} - n_{h,t-1} = f_h(\tilde{W}_{h,t}', n_{h,t-1}, \tilde{W}_{-h,t}, n_{-h,t-1}, \xi_{t-1}) - f_h(\tilde{W}_{h,t}', n_{h,t-1}, \tilde{W}_{-h,t}, n_{-h,t-1}, \xi_{t-1})$. From the Lipschitz continuity of $f_h$

$$-K|\tilde{W}_{h,t} - \tilde{W}_{h,t}'| \leq f_h(\tilde{W}_{h,t}, \ldots) - f_h(\tilde{W}_{h,t}', \ldots) \leq K|\tilde{W}_{h,t} - \tilde{W}_{h,t}'|,$$

where $K$ is a constant from (A1). Thus there exists a value $k_{h,t} \in [-K, K]$ such that $f_h(\tilde{W}_{h,t}, \ldots) - f_h(\tilde{W}_{h,t}', \ldots) = k_{h,t}|\tilde{W}_{h,t} - \tilde{W}_{h,t}'|$. Plugging into (12) we get

$$R\Delta p_t = \sum_h \rho_h(n_{h,t} - n_{h,t-1}) = |\tilde{R}_t| \sum_h \rho_h k_{h,t}|z_{h,t-1} - z_{h,t-1}'|.$$

Denote $p_t + y - R p_t$ as $b_t$. Notice that $b_t = 0$ if and only if $p_t$ is equal to the risk-neutral fundamental price, and $b_t > 0 \iff p_t < p_n$. It can be immediately verified that $\tilde{R}_t - b_t = R\Delta p_t$ and so (15) is equivalent to

$$22$$
\[
\tilde{R}_t \left(1 - \text{sgn}(\tilde{R}_t) \sum_h \rho_h k_{h,t} |z_{h,t-1} - z'_{h,t-1}| \right) = b_t,
\]
(16)
where \(\text{sgn}(\tilde{R}_t)\) is the sign function. The derivation of (16) was the goal of Step 2.

**Step 3:** The expression inside the parenthesis in (16) is positive.

\[
\left| \text{sgn}(\tilde{R}_t) \sum_h \rho_h k_{h,t} \cdot |z_{h,t-1} - z'_{h,t-1}| \right| \leq \sum_h \rho_h |k_{h,t}| \cdot |z_{h,t-1} - z'_{h,t-1}| \leq K \rho_H^2 H < 1.
\]

In the derivation above, inequality (a) follows from (3), the definition of \(z'_{h,t-1}\) and from the fact that \(|k_{h,t}| \leq K\). Inequality (b) follows from the fact that if \(\tilde{W}_{min} \leq \tilde{W}_{h,t} \leq \tilde{W}_{max}\) then \(\rho_1 \leq \rho'_{h} \leq \rho_H\). Finally the last equality follows from (A4).

**Step 4:** The sequence of market-clearing prices is monotone.

\[
\tilde{R}_t \begin{cases} \geq 0 \iff b_t \geq 0 \iff p_n \geq p_t. 
\end{cases}
\]
(17)
From (17) and from Lemma 34 we have that

\[
\tilde{R}_t > 0 \Rightarrow (\Delta p_t \geq 0, p_n > p_t) \Rightarrow p_n > p_{t-1},
\]
\[
\tilde{R}_t < 0 \Rightarrow (\Delta p_t \leq 0, p_n < p_t) \Rightarrow p_n < p_{t-1},
\]
\[
\tilde{R}_t = 0 \Rightarrow (\Delta p_t = 0, p_n = p_t) \Rightarrow p_n = p_{t-1},
\]
which means that

\[
\tilde{R}_t \begin{cases} \geq 0 \iff p_n \geq p_{t-1}. 
\end{cases}
\]
Combining this we immediately get monotonicity. Indeed, if, say, \(p_n > p_{t-1}\) then \(\tilde{R}_t > 0\) and so \(\Delta p_t = p_t - p_{t-1} \geq 0\) and \(p_n > p_t\). Applying the same logic we can get that \(p_{t+1} \geq p_t\) and \(p_n > p_{t+1}\) and so on. Thus the price sequence is increasing. Similarly, it can be shown that if \(p_{t-1} > p_n\) then the price sequence is decreasing.

The boundedness of the price sequence together with Step 4 implies that it converges which completes the proof of the lemma.
Lemma 3.6. Take any equilibrium path \( \{(n_t, p_t)\} \) and consider sequence \( \{\tilde{R}_t\} \) along this path:

i) If there exists period \( T^0 \) and a constant \( c > 0 \) such that either \( \tilde{R}_t > c > 0 \) for any \( t \geq T^0 \) or \( \tilde{R}_t < -c < 0 \) for any \( t \geq T^0 \) then only one type survives in the limit.

ii) If \( \tilde{R}_t \to \tilde{R}_\infty \) and \( \tilde{R}_\infty \neq 0 \) then only one type survives in the limit.

\[ \triangleright \]

We will prove part i) only for the case of an eventually positive sequence of \( \tilde{R}_t \). The case of eventually negative sequence is similar. Part ii) is an immediate corollary of part i) and the proof is not provided.

**Step 1.** Let \( S \) be a compact subset of the following set:

\[ S \subset \{ (n, W, \xi) \mid \exists h, h' : n_h > 0, n_{h'} > 0 \& W_h < W_{h'}; n \in \Delta^{H-1}; \xi \in \Xi \} . \]

Define \( I(n, W, \xi) \) as

\[ I(n, W, \xi) = \min_{\{h: f_h(n, W, \xi) > n_h\}} [f_h(n, W, \xi) - n_h], \]

where \( \min \) over the empty set is defined to be zero. Then it must be the case that

\[ \inf_{\tilde{S}} I(n, W, \xi) = \min_{\tilde{S}} I(n, W, \xi) > 0. \]

\[ \triangleright \]

Function \( I(n, W, \xi) \) compares corresponding components of the vector of shares \( n \) with the vector of new shares \( f(n, W, \xi) \). It is equal to the lowest positive increase in the shares, or zero if there is none. By definition \( I(n, W, \xi) \geq 0 \). However, if \( (n, W, \xi) \in S \) then by the non-triviality axiom \( I(n, W, \xi) > 0 \). Set \( S \) is compact and so \( \inf_{\tilde{S}} I(n, W, \xi) = \min_{\tilde{S}} I(n, W, \xi) > 0 \), because the infimum is reached at some point in \( S \), and at this point \( I(n, W, \xi) > 0. \)

**Step 2:** There is a type \( P \leq H \) such that \( n_{P, t} \) converges to a positive limit. Furthermore, all types with higher beliefs have zero shares for any \( t \geq T^0 \).

\[ \triangleright \]

Consider the most optimistic type \( H \). When \( t \geq T^0 \) this type will earn the highest profit. By weak monotonicity \( n_{H, t} \leq n_{H, t+1} \leq \ldots \) and so \( n_{H, t} \) converges. Thus either for any \( t \geq T^0 \) the share of \( H \) is equal to zero, or \( n_{H, t} \) forms an increasing sequence that converges to a positive limit \( n_H > 0 \).

In the latter case we are done. In the former case, take type \( H - 1 \). Using the same argument we can argue that if this type has a positive weight at some moment after \( T^0 \) then by weak monotonicity its shares will form a weakly increasing sequence that converges to a positive limit. Alternatively, \( n_{H-1, t} = 0 \) for all \( t \geq T^0 \), in which case we move to type \( H - 2 \) and so on. \[ \triangleright \]

Now we can complete the proof of the lemma. Let \( P \) be the index of the type from Step 2. We know that its shares converge to a positive limit \( n_P > 0 \). If \( n_P = 1 \) then the shares of
other types will converge to zero and we are done. Suppose now that \( n_P < 1 \) and show that this case is impossible.

Since \( \tilde{R}_t \) is eventually positive and is bounded away from zero

\[
\exists \gamma > 0 \forall t \geq T^0 \forall h, h' : |\tilde{W}_{h',t} - \tilde{W}_{h,t}| \geq \gamma.
\]  

(18)

By definition of the equilibrium, the set \([1/R(\rho_1 - \sigma^2 S), 1/R(\rho_H - \sigma^2 S)]\) contains all equilibrium price sequences, \( \{p_t\} \). Therefore, there is a compact set \( W \) that will contain all equilibrium wealth sequences, \( \{\tilde{W}_t\} \). Define \( S' \) as

\[
S' = \left\{(n, W, \xi) \mid 0 < \alpha \leq n_P \leq \beta < 1; \forall h, h' : |\tilde{W}_{h',t} - \tilde{W}_{h,t}| \geq \gamma; n \in \Delta^{H-1}; W \in W; \xi \in \Xi \right\},
\]

where \( \alpha \) and \( \beta \) are two constants that separate \( n_P \) from zero and one. Two things can be noticed about \( S' \). First, after \( T^0 \), the equilibrium sequence \( \{(n_t, \tilde{W}_t, \xi_{t-1})\} \) will belong to \( S' \). Second, Step 1 can be applied to \( S' \). Indeed, \( S' \) is a compact set and for any point in \( S' \) there are at least two types with positive shares (one of them is type \( P \)) and different wealths. From Step 1 we know that the smallest positive increase in shares is bounded away from zero. But then it is impossible that \( n_{P,t} \) converges. This is a contradiction and thus \( n_{P,t} \) must converge to one. ▶

**Theorem 4.11:** Assume that beliefs are diverse and that shares of all types are positive in \( t = 1 \). Assume also that \( \varepsilon_t \) is distributed with support \([-M; M]\). Then with probability one there will be a time \( T \leq \infty \) such that for any \( t \geq T \)

\[
p_n - \frac{M}{R - 1} \leq p_t \leq p_n + \frac{M}{R - 1}.
\]

▶ First, notice that Lemma 3.4 that was proved for the deterministic case is still valid. Its proof does not change as long as it is understood that \( R_t = p_t + y + \varepsilon_t - R_{p_{t-1}} \) should be used instead of \( \tilde{R}_t \). Lemma 3.4 in its stochastic version says that \( R_t \) and \( \Delta p_t \) have the same sign unless \( n_{h,t} = n_{h,t-1} = 1 \) for some \( h \). In the latter case \( \Delta p_t = 0 \) regardless of \( R_t \).

Second, by following the same steps that we used to derive (15) in Lemma 3.5 we can see that:

\[
R\Delta p_t = |R_t| \sum_h \rho_h k_{h,t} |z_{h,t-1} - z'_{h,t-1}| = R_t \sum_h \rho_h k_{h,t} |z_{h,t-1} - z'_{h,t-1}| \cdot \text{sgn}(R_t)
\]

(19)

Variables \( k_{h,t} \) and \( z'_{h,t-1} \) have precisely the same meaning as in Lemma 3.5 and \( \text{sgn}(R_t) \) is a standard sign function. Denote the multiplier of \( R_t \) in (19) as \( A_t \) and re-write (19) as

\[
R\Delta p_t = R_t A_t.
\]

(20)
It is easy to see that \( 0 \leq A_t < 1 \). The first inequality follows from the stochastic version of Lemma 3\( i \) if there is no \( h \) so that \( n_{h,t} = n_{h,t-1} = 1 \) then \( R_t \) and \( \Delta p_t \) have the same signs and then from (20), \( A_t > 0 \). If \( n_{h,t} = n_{h,t-1} = 1 \) for some \( h \) then \( \Delta p_t = 0 \) since the distribution of beliefs did not change. In this case if \( R_t \neq 0 \) then \( A_t = 0 \) from (20). If \( R_t = 0 \) then \( A_t = 0 \) by definition. As for the inequality \( A_t < 1 \) it can be proved by using the same logic as in Step 3 of Lemma 3\( i \).

As before, let \( b_t \) denote \( p_t + y - Rp_t \) and notice that \( y \) does not include a random term. With this definition, \( b_t = 0 \) if and only if \( p_t = p_n \) and \( b_t > 0 \) implies that \( p_t < p_n \). When using \( b_t \) instead of \( p_t \) expression (13) in the statement of the theorem 4\( i \) becomes
\[
-M \leq b_t \leq M. \tag{21}
\]
To prove theorem 4\( i \) we need to show that \( \{b_t\} \) will reach \([−M; M]\) — either in finite time or in the limit — and will remain inside this interval.

We apply some algebra to transform (20). It is easy to see that \( R_t = b_{t-1} + \Delta p_t + \varepsilon_t \), and so (20) can be re-written as
\[
(R - A_t)p_t = (R - A_t)p_{t-1} + A_t b_{t-1} + A_t \varepsilon_t.
\]
After multiplying the equality above by \((1 - R)\) and adding \((R - A_t)y\) to both sides we have that
\[
(R - A_t)b_t = R(1 - A_t)b_{t-1} - (R - 1)A_t \varepsilon_t.
\]
Thus
\[
b_t = \frac{R(1 - A_t)}{R - A_t} b_{t-1} - \frac{(R - 1)A_t}{R - A_t} \varepsilon_t, \tag{22}
\]
and so
\[
\Delta b_t = -\frac{(R - 1)A_t}{R - A_t} (b_{t-1} + \varepsilon_t). \tag{23}
\]

**Step 1.** If \( |b_{t-1}| \leq M \) then \( |b_t| \leq M \) with probability 1, and thus \( |b_t| \leq M \) is an absorbing range.

\( \blacksquare \) This immediately follows from (22). If \( |b_{t-1}| \leq M \) then the highest possible value of \( b_t \) is when \( b_{t-1} \) is equal to \( M \) and \( \varepsilon_t \) is equal to \(-M\). Then \( b_t \) is equal \( M \). The lowest possible value of \( b_t \) is reached when \( b_{t-1} \) is equal to \(-M\), and \( \varepsilon_t \) is equal to \( M \), and then \( b_t \) is equal to \(-M\), which proves step 1. \( \blacksquare \)

**Step 2.** If \( b_{t-1} > M \) then with probability one \( b_{t-1} \geq b_t \) and \( b_t > -M \). If \( b_{t-1} < -M \) then with probability one \( b_{t-1} \leq b_t \), and \( b_t < M \).

\( \blacksquare \) Recall that \( 0 \leq A_t < 1 \), and so the fraction in (23) is non-negative. Therefore from (23) it follows that if \( b_{t-1} < -M \) then \( \Delta b_t \) is non-negative and so \( b_t \geq b_{t-1} \). The fact that \( b_t < M \) follows immediately from (22). The case of \( b_{t-1} > M \) is similar.\( \blacksquare \)
An immediate corollary of Step 2 is that \( b_{t-1} \) and \( b_t \) cannot be on different sides of the absorbing range \([-M, M]\). That is, if \( b_{t-1} < -M \) then \( b_t < M \) and if \( b_{t-1} > M \) then \( b_t > -M \). Combining Steps 1 and 2 we have the following: if there is a \( T' < \infty \) such that \( |b_T| \leq M \) then \( |b_t| \leq M \) for any \( t > T' \). If such \( T' \) does not exist then either \( b_t > M \) for all \( t \) or \( b_t < -M \) for all \( t \).

If such \( T' \) exists we are done. If it does not then the next step shows that \( b_t \to b \) and \( |b| = M \). Thus the absorbing range will be reached in the limit.

**Step 3.** If \( |b_t| > M \) for all \( t \) then \( b_t \to b \) and \( |b| = M \).

\[\text{Consider the case when } b_t > M \text{ for any } t. \text{ By Step 2 the } b\text{-sequence is monotone and since it is bounded it converges. Denote the limit as } b \text{ and assume that } b > M. \text{ Given that } \{b_t\} \text{ converges, the price sequence also converges and thus } R\Delta p_t \to 0. \text{ Therefore, by taking the algebraic identity } R_t = b_t + R\Delta p_t + \varepsilon_t \text{ to the limit we have that there exists } T^0 \text{ and } c > 0 \text{ such that } R_t > c > 0 \text{ for any } t > T^0 \text{ (recall that } b > M).\]

Lemma 3.6 part i) showed that if \( \tilde{R}_t \) eventually becomes positive (or negative) and is bounded away from zero then only one type survives in the long-run. While the lemma itself was proved in the deterministic case it remains valid in the stochastic case as well. The proof remains the same except that \( R_t \) should be used instead of \( \tilde{R}_t \). Using the stochastic counterpart of Lemma 3.6 we can conclude, therefore, that if \( \{b_t\} \) does not reach the absorbing range then only one type survives in the long-run. Since \( b_1 > M \), it will be the most optimistic type. Indeed, by weak monotonicity \( n_{H,1} \leq n_{H,2} \leq \cdots \) and so \( \{n_{H,t}\} \) is an increasing sequence. Since \( n_{H,1} > 0 \) its limit is positive and thus it has to be equal to one. However, this is a contradiction, since if it is the most optimistic type that survived, it would mean that \( p_t \) will be eventually greater than or equal to \( p_n \) which is a contradiction to \( b > M \).

This completes the proof of the theorem.
References


