We present an integral equation approach for the valuation of American-style derivatives when the underlying asset price follows a general diffusion process and the interest rate is stochastic. Our contribution is fourfold. First, we show that the exercise region is determined by a single exercise boundary under very general conditions on the interest rate and the dividend yield. Second, based on this result, we derive a recursive integral equation for the exercise boundary and provide a parametric representation of the American option price. Third, we apply the results to models with stochastic volatility or stochastic interest rate, and to American bond options in one-factor models. For the cases studied, explicit parametric valuation formulas are obtained. Finally, we extend results on American capped options to general diffusion prices. Numerical schemes based on approximations of the optimal stopping time (such as approximations based on a lower bound, or on a combination of lower and upper bounds) are shown to be valid in this context.

(American Options; Valuation; Optimal Exercise; Diffusion; Stochastic Interest Rate; Stochastic Volatility; Integral Equation; Capped Options; Bounds and Approximations)

1. Introduction

The valuation and hedging of American-style derivatives is a challenge for both academia and industry. Indeed, to value such claims it is necessary to identify the optimal exercise region (i.e., the set of prices and times at which it is optimal to exercise the contract). This inherent difficulty in the valuation process has prompted considerable attention to simple settings in which the underlying asset price follows a lognormal process and the interest rate is constant (i.e., the standard model, or Black-Scholes setting). In this context, Kim (1990), Jacka (1991), and Carr et al. (1992) have established that the American option price is equal to the corresponding European option price plus an Early Exercise Premium (EEP) which captures the benefits from exercising prior to maturity.1 This EEP representation of the option price has proved extremely useful in that it provides a recursive integral equation for the optimal exercise boundary. Solving the integral equation resolves the valuation problem: It identifies the exercise boundary and produces a parametric formula for the American option price. This approach, which is based on the integral equation, is easy to implement and has significant advantages over other numerical procedures such as methods based on Partial Differential Equations (PDE), Monte Carlo simulation, or binomial lattices.2 Moreover, it can be implemented in a number

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1 See also Jamshidian (1992) for American bond options and Gao et al. (2000) for American barrier options.
2 Huang et al. (1996) motivates the integral equation method. See also Broadie and Detemple (1996) and Ju (1998) for comparisons
of ways because several numerical schemes have been developed to compute the solution of the integral equation.³

Resolutions of the American option valuation problem in the standard model, however, are confronted with abundant empirical evidence suggesting that stock prices do not satisfy the lognormality assumption embedded in the model. The “volatility smile” phenomenon is a well-known consequence of this discrepancy.⁴ For consistency with empirical regularities more elaborate models, which allow for stochastic volatility, interest rate, and dividend yield, are needed. Unfortunately, there are significant difficulties in extending the aforementioned analytic results to the case of general diffusion prices and stochastic interest rate.⁵ Even though typical present value representations of the American option price, such as the EEP formula, are valid with diffusions, little is known about the exercise region even for a deterministic interest rate. Some insights can be gathered from Bergman et al. (1996), who deal with European-style contingent claims with convex payoff functions, and establish the convexity of the valuation formulas. Similar intuitions emerge from El Karoui et al. (1998), who examine the hedging of American-style claims when volatility is misspecified and the interest rate is constant. One of their key results also establishes that the value of a European- or American-style claim with a convex payoff is a convex function of the underlying asset price. The same result is obtained by Hobson (1998), using a different method. For deterministic interest rate this “convexity” result implies that the time section of the exercise region is convex, and hence it is a finite or infinite interval. Perhaps the most precise result to date about the structure of the exercise region can be found in Jacka and Lynn (1992), who consider general contingent claims written on diffusion price processes. For the one-dimensional case they show up-connectedness of the exercise region if the drift of the payoff process (i.e., the negative of the instantaneous exercise premium) is decreasing in the underlying price at all times and for all possible values of the asset price.

This article provides a systematic treatment of American options when the volatility of the underlying asset return, its dividend yield, and the interest rate are stochastic. Although our focus is on options, our method is general and applies to more complex derivatives. Because the exercise region/boundary is a critical ingredient in the valuation and hedging of American-style claims, we place special emphasis on its determination. Our first contribution is to identify the “geometric structure” of the American call option’s exercise region. As mentioned above, for deterministic interest rate the time section of the exercise region is a convex set, and hence it is a finite or infinite interval. This leaves open the possibility of two exercise boundaries (upper and lower). We provide simple conditions under which the time section of the exercise region is up-connected, that is, an infinite interval. Consequently, the exercise region is determined by a single exercise boundary curve. This property holds in a large class of models with stochastic volatility and mild restrictions on the interest rate and the dividend specification. It holds even in diffusion models with multiple state variables, under suitable assumptions. In the particular case of deterministic interest rate, we can further prove that the unique boundary curve is continuous, as was shown by Jacka (1991) in the Black-Scholes setting. Moreover, for the one-dimensional case, our conditions are distinct from those in Jacka and Lynn (1992): They do not imply, nor are they implied, by their conditions.

Once the geometry of the exercise set is known the characterization of the exercise boundary becomes easy. General results on the EEP representation can be applied and specialized to derive a recursive integral equation for the boundary. This equation can be
used as a starting point for numerical methods, as was done in the log-normal case.

Next we turn to applications. American options are valued when the underlying price follows a Constant Elasticity of Variance (CEV) process. A model with stochastic interest rate is also examined. American bond options are studied in a quadratic model of the term structure. In all these cases explicit parametric formulas are obtained for the integral equation characterizing the exercise boundary, and for the option price. As a result, numerical methods employed for the log-normal case are also valid in these contexts.

Our last contribution is to examine the connection between the exercise regions of various American option contracts with related payoff functions. Simple relationships between exercise regions often provide quick and useful information, in particular when pricing involves time-consuming numerical computations. For instance, it is natural to ask whether the exercise region of an exotic option can be recovered from the exercise region of a plain vanilla option. Broadie and Detemple (1995, 1996) first discovered a simple relation between the exercise regions of American capped call options and of standard American call options for the lognormal process. With the help of this result, they developed efficient methods (the Lower Bound Approximation (LBA) and the Lower-Upper Bound Approximation (LUBA)) to value American options. We show that these relations extend to general diffusion processes. For constant caps, immediate exercise is optimal at the first time at which the asset price reaches the minimum of the cap and the exercise boundary of the corresponding uncapped option. Even more strikingly, we show that the exercise region of options with growing caps can be described by a three-parameter policy for any diffusion price process. Moreover, we establish the validity of their approximation methods in the context of general diffusion processes.

The layout of the paper is as follows. Section 2 discusses the approach to American option valuation when the underlying asset price follows a diffusion process. The optimal exercise boundary is characterized for single as well as multiple factors. Section 3 presents an application to a stochastic volatility model, the CEV model. Section 4 presents applications to the valuation of American bond options; again, an explicit formula is obtained for quadratic term structure models. Section 5 deals with stochastic interest rates, such as the Hull-White term structure model. In §6 we examine the relation between the exercise regions of American capped call options and of standard American call options for general diffusion processes. Numerical methods are discussed in §7 and are used throughout the paper to illustrate the properties of option prices in the various models studied. Conclusions are formulated in §8. All proofs are collected in the Appendix and a companion Technical Appendix (available as an electronic companion on the Management Science website at <mansci.pubs.informs.org/>)

2. American Call Options
We consider an economy in which the uncertainty is represented by a standard Brownian motion process. The risk-neutralized underlying asset price follows a diffusion process

\[ dS_t/S_t = (r(S_t, t) - \delta(S_t, t)) dt + \sigma(S_t, t) dW_t, \]  

(1)

where \( r(S, t) \) denotes the interest rate, \( \sigma(S, t) \) the volatility, and \( \delta(S, t) \) the dividend yield. We assume that the coefficients \( r(S, t), \sigma(S, t), \delta(S, t) \) are continuously differentiable with bounded derivatives, and also that \( \sigma(S, t) \) is positive almost everywhere.

\[6\] American option prices with stochastic interest rate are studied in Amin and Bodurtha (1995), Chung (1999), Ho et al. (1997), and Menkveld and Vorst (2000). These papers employ extrapolation schemes (e.g., Geske and Johnson 1984, Bunch and Johnson 1992), and do calculations based on a small set of exercise points. The shortcomings of this method are well understood (see Huang et al. 1996). Also, results on the exercise boundary and the exercise premium are absent from this literature.

\[7\] The valuation of American bond options has been explored in some popular term structure models (e.g., Jamshidian 1992, Chesney et al. 1993). This literature, typically, assumes properties of the exercise set to characterize the exercise boundary.

\[8\] This model can be used to value American-style currency options, commodity contracts, or index options. In these applications the dividend rate \( \delta \) might represent the foreign risk-free interest rate, or an instantaneous proportional cost of carry or a convenience yield.
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Moreover, we assume that \( r(S, t) \) is a decreasing function of \( S \), and an increasing function of \( t \), and \( \delta(S, t)S \) is an increasing function of \( S \). This last set of assumptions captures the intuitive notion that the interest rate is high when asset prices are low and that dividends are high when asset prices are high.\(^9\) We also assume that (1) satisfies a monotonicity condition with respect to time shifts in the coefficients. Specifically, for \( s \in [0, T] \) and \( h \geq 0 \), let \( S^{h,s} \) be the solution of (1) with initial condition \( S_s = S \), at time \( s \), and time-translated parameters \( r(S, t+h), \delta(S, t+h) \) and \( \sigma(S, t+h) \). Then, we require that \( S^{h,s}_0 \geq S^{h,v}_0 \) for all \( h \geq 0 \) and \( s, v \in [0, T] \) with \( v \geq s \). This time-monotonicity condition, along with the assumptions on the interest rate function, ensure that the prospects of the call option holder do not improve with the passage of time. The condition is automatically satisfied when the coefficients of the model are independent of time. The evolution of \( S \) in (1) is represented under the risk-neutral measure \( Q \): \( W(t) \) is a \( Q \)-Brownian motion process. Note that the dividend yield, the interest rate, and the volatility may depend on the asset price and therefore evolve stochastically over time. Because frictionless trading in the risky and riskless assets is permitted, the economy has complete markets.

Following Harrison and Pliska (1981), any derivative can be uniquely priced as the expected value of its discounted future payoff where the expectation is taken under the risk-neutral measure. By Fakeev (1971), the price of an American derivative is given by the Snell envelope of the expected value process under some technical conditions on the payoff function. In particular, the price \( C(S, t) \) of an American call option written on the asset price \( S \) and with maturity date \( T \) and strike price \( K \) is

\[
C(S, t) = \sup_{\tau \in \mathcal{F}_{i,T}} \mathbb{E}_t^* \left[ \exp \left( - \int_t^\tau r(S_v, v) dv \right) \max(S_v - K, 0) \right],
\]

where \( E_t^* \) denotes the conditional expectation, at date \( t \), under the risk-neutral measure, and \( \mathcal{F}_{i,T} \) is the set of stopping times of the Brownian filtration.

Let \( \mathcal{E} = \{(S, t) \in \mathbb{R}_+ \times [0, T] : C(S, t) = \max(S - K, 0)\} \) denote the immediate (optimal) exercise region of the call contract. Our first proposition records a useful property of this region.

**Proposition 1.** The exercise region of the American call option with maturity \( T \) and strike \( K \) is \( \mathcal{E} = \{(S, t) \in \mathbb{R}_+ \times [0, T] : S \geq B(t)\} \) for a nonincreasing right-continuous function \( t \to B(t) \). Moreover, \( B(t) \) is continuous if the interest rate is deterministic.

This proposition identifies interesting features of the exercise region. First, it states that immediate exercise is optimal whenever the underlying asset price exceeds a boundary \( B \) which depends on time alone. At first sight this may appear surprising because the coefficients of the financial market model are stochastic. It is, however, quite natural since the only source of uncertainty is embedded in the (Markovian) underlying asset price. Naturally, the critical asset price depends on the parameters of the model, including the structure of the asset price and of the interest rate. Second, it also establishes that the exercise region is up-connected and that the boundary is unique. This property is not evident when the coefficients depend on the underlying price. Third, the boundary is right continuous: Unlike the traditional case of a lognormal underlying asset price process, it is difficult to show continuity. This is also due to the dependence of the model's coefficients on the asset price.

Proposition 1 is closely related to Theorem 4.3 in Jacka and Lynn (1992), which provides conditions for up- (or down-) connectedness of the exercise region in a general diffusion setting such as (1). In essence they assume a smooth payoff function \( g(S, t) \) and show that up-connectedness holds if the Black-Scholes operator \( \mathcal{L}g(S, s) \) decreases in \( S \) for all \( (S, s) \in \mathbb{R}_+ \times [0, T] \). Our result differs in that it (i) does not require smoothness of the payoff function over the whole domain \( \mathbb{R}_+ \times [0, T] \) and (ii) applies to payoffs which depend on the history of the underlying price, i.e., \( g(S, \cdot, s) = \)

\footnotesize
\(^9\)The dependence of the interest rate \( r(S, t) \) on the asset price \( S \) enables us to deal with index options. This dependence is justified by a general equilibrium analysis which shows that the market value of the index and the interest rate are endogenous functions of a common underlying variable, the aggregate dividend.

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exp\left[-\int_0^t r(S_u, v) \, dv\right] (S_t - K)^+.  \tag{10}

It is similar in that

exp\left[-\int_0^t r(S_u, v) \, dv\right] [r(S_t, s) K - \delta(S_t, s) S_t] on the event \{(S_t, s) \in \mathbb{R}^+\}

which is a non-increasing functional of the trajectories of $S$. This behavior of the operator $\mathcal{L}g(S_t, S_{\{\cdot\}}, s)$ over the exercise region (or a set containing the immediate exercise region) is sufficient for the validity of the property.  \footnote{It is possible to expand the state space to write the problem in terms of a vector diffusion. Jacka and Lynn (1992) do not study the exercise region in this multidimensional setting.}

As stated in Remark 1 in the appendix, our result extends to payoffs that are more general than options' payoffs. In this context up-connectedness holds at a given time $t$ under the general monotonicity property (A8). This condition does not imply that $\mathcal{L}g(S_t, s)$ is nonincreasing in $S$ over the whole domain of $(S_t, s)$.

With the help of Proposition 1 we can apply and specialize general results about the Early Exercise Premium (EEP) representation (Rutkowski 1994). By the general EEP theorem, the American call option price can be divided into two parts. The first component is the value of a European call option with the same characteristics (maturity and strike). The second component is the early exercise premium, which captures the benefits from exercise prior to maturity. These are simply the dividend collected on the underlying asset net of the interest cost on the cash (strike) given up to exercise the option. In other words, an American option is equal to a portfolio of a European option and the value of a continuous cash flow which rewards early exercise. The results of Proposition 1 enable us to write the exercise premium in a simple manner in terms of the cash flows generated when the asset price exceeds the unique exercise boundary.

Specifically, the exercise premium can be expressed as an integral of terms which depend on the exercise boundary $B(t)$ as follows:

$$C(S_t, t; B(\cdot)) = C'(S_t, t) + \Pi(S_t, t; B(\cdot)),$$  \tag{3}

where

$$\Pi(S_t, t; B(\cdot)) = \mathbb{E}_t \left[ \int_t^T e^{-\int_t^v r(S_u, v) \, dv} (\delta(S_v, v) S_v - r(S_v, v) K) 1_{\{S_v \geq B(v)\}} \, dv \right]$$  \tag{4}

is the early exercise premium, $C'(S_t, t)$ the European option price, and $1_A$ the indicator of the set $A$. The second term indicates that the incremental gain over the time period $[t, t + dt]$ from exercising the option at time $t$ is $(S(S_t, t) S_t - r(S_t, t) K) dt$. Furthermore, because immediate exercise is optimal when $S_t = B(t)$, we obtain the following recursive integral equation for the exercise boundary $B(t)$:

$$B(t) - K = C'(B(t), t) + \Pi(B(t), t; B(\cdot)),$$  \tag{5}

subject to the boundary condition $B(T_\cdot) = \max\{K, r(T, B(T_\cdot))/\delta(T, B(T_\cdot)) K\}$, where $B(T_\cdot) = \lim_{t \to T} B(t)$. Because the interest rate and the dividend yield depend on the underlying price, the boundary at the maturity date solves a nonlinear equation.  \footnote{The exercise boundary and the valuation formula for an American put option can be obtained from the call formulas by a change of numeraire. This relation, known as put-call symmetry, was demonstrated in a general context by Schroder (1999). A review of the literature and issues surrounding put-call symmetry can be found in Detemple (2001).}

### 2.1. The Exercise Boundary in a Multivariate Model

In the model of Proposition 1 all the uncertainty is summarized in a single factor, the underlying asset price. This is overly restrictive because it implies perfect correlations between the asset return on the one hand and the interest rate, the dividend rate, or the asset volatility on the other hand. Imperfect correlation between these variables is a well-established empirical regularity. Models with multiple state variables are therefore of theoretical as well as practical interest.
Accordingly, suppose that the risk-neutralized evolution of the asset price $S$ is given by

$$\frac{dS_t}{S_t} = [r(S_t, Y_t, t) - \delta(S_t, Y_t, t)]dt + \sigma(S_t, Y_t, t)dW_t, \quad (6)$$

$$dY_t = [\alpha(S_t, Y_t, t) - \gamma(S_t, Y_t, t)\theta(S_t, Y_t, t)]dt + \gamma(S_t, Y_t, t)dW_t, \quad (7)$$

where $Y$ represents a vector of factors (state variables), $W$ is a $d$-dimensional Brownian motion process, and $\theta$ is the market price of $W$-risk. We assume that the coefficients are continuously differentiable functions with bounded derivatives. In this setting the state variables $Y$ affect the evolution of $S$ through the interest rate, the dividend rate, and the asset price volatility. Conversely, the asset price $S$ influences the dynamics of the state variables $Y$. If the previous assumptions do not imply monotonicity, then they are amended as follows:

**Assumption M.**

(i) Trajectories of the stock price process are non-decreasing with respect to the initial stock price; i.e., $S_t \geq S_v$ implies $S_v \geq S_0$ for all $v \geq t$, and all $t \in [0, T]$.

(ii) Trajectories of the interest rate process are non-increasing with respect to the initial value of the stock price; i.e., $S_t \geq S_v$ and $Y_t = Y_v$ implies $r(S_t, Y_v, v) \leq r(S_v, Y_v, v)$ for all $v \geq t$, and all $t \in [0, T]$.

(iii) Trajectories of the dividend process are non-decreasing with respect to the initial value of the stock price; i.e., $S_t \geq S_v$ and $Y_t = Y_v$ implies $\delta(S_t, Y_v, v)S_v \geq \delta(S_v, Y_v, v)S_t$ for all $v \geq t$, and all $t \in [0, T]$.

In essence, Assumption M imposes monotonicity conditions with respect to the initial asset price. When the asset price and the state variables satisfy the joint diffusion (6)-(7), perturbations in initial conditions have direct as well as indirect effects on the paths of $S$. The direct effects result from the impact of $S$ on the coefficients $r(S_t, Y_t, t)$, $\delta(S_t, Y_t, t)$, and $\sigma(S_t, Y_t, t)$. Indirect effects are caused by the perturbations in the trajectories of $Y$, which induce a further variation in $r(S_t, Y_t, t)$, $\delta(S_t, Y_t, t)$, and $\sigma(S_t, Y_t, t)$. Condition (i) ensures that the paths of $S$ are nondecreasing in the initial conditions ($P-a.s.$).

The motivation for condition (ii) is similar: Because of indirect effects through $Y$, the condition that $r(S, Y, v)$ is decreasing in the first argument $S$ does not guarantee that the paths $\{r(S_v, Y_v, v) : v \geq t\}$ will be nonincreasing. The rationale for (iii) is identical.

For the one-dimensional case these conditions reduce to those of the prior section. The Comparison Theorem for solutions of stochastic differential equations (Karatzas and Shreve 1988, p. 293) also ensures that condition (i) is automatically satisfied when the coefficients of the state variables process (7) are independent of $S$ (but depend on $Y$). In this instance the trajectories of the state variables $Y$ are not affected by a perturbation of the initial asset price. Indirect effects on the coefficients of (6) vanish; in a sense we are back to a one-dimensional equation, albeit with random coefficients.

The price specification (6)-(7) is very general and covers many special cases of interest. For instance, in the typical stochastic volatility model, both $r$ and $\delta$ are constant, while $\sigma(S, Y, t) = Y$; i.e., $Y$ represents volatility. In models with stochastic interest rate, both $\delta$ and $\sigma$ are typically constant; in one-factor models $r = r(Y, t)$ is taken as a function of a state variable $Y$. In both classes of models the coefficients of the state variable process (7) are independent of the price $S$.

Consider now the American call option with maturity $T$ and strike $K$. By the Markovian property of the state variables $S$ and $Y$, the time $t$ price of this option is a function $C(S, Y, t)$ of $S$, $Y$, and $t$, where $S$ and $Y$ are the values at time $t$. Given the generality of this model it would seem difficult to identify general properties of the exercise region $g = \{(S, Y, t) \in \mathbb{R}_+ \times \mathbb{R} \times [0, T] : C(S, t) = \max(S - K, 0)\}$ in this setting. Somewhat surprisingly, a version of Proposition 1 still applies.

**Proposition 2.** Consider the asset price specification $(S, Y)$ in (6)-(7) and suppose that Assumption M holds. The exercise region of the American call option with maturity $T$ and strike $K$ is $g = \{(S, Y, t) \in \mathbb{R}_+ \times \mathbb{R} \times [0, T] : S \geq B(Y, t)\}$, for a function $B(Y, t)$.
This proposition establishes that immediate exercise is optimal whenever the underlying asset price exceeds a boundary surface \( B(Y, t) \), which depends on time and on the state variables. The property may appear surprising since the coefficients of the financial market model now depend on both the asset price and the state variables. The source of the result is again the Markovian structure of the environment, which implies that the pair \((S, Y)\) provides a complete description of the state at date \( t \) and hence of the evolution of prices. The critical exercise point at date \( t \) will then depend on the value \( Y \).

The up-connectedness of the exercise region and the uniqueness of the boundary surface are perhaps even more striking. These properties are even less evident when the coefficients depend on \((S, Y)\). A key ingredient here is the Monotonicity Assumption M. Suppose \((S, Y, t) \in \mathcal{E}\) and consider an increase, by \( \Delta S \), of the asset price \( S \) at date \( t \). In combination with the boundedness (by one) of the slope of the call payoff the conditions of Assumption M ensure that the gain in the value of any exercise policy is bounded above by \( \Delta S \) (i.e., \( \Delta C \leq \Delta S \)). Since immediate exercise provides a gain exactly equal to the increase in \( S \) (i.e., \( \Delta C = \Delta S \)) when \((S, Y, t) \in \mathcal{E}\) and is a feasible strategy, it dominates any waiting policy. In other words, as in the previous model, the slope of the option price is bounded above by one (in fact it equals one) in the exercise region.

Alternatively, we can think in terms of the local cost of delaying exercise, which equals the dividend collected net of the interest loss on the strike. For the call payoff the cost is \( \Phi(S_t, Y, t) = \exp[-\int_{(S_t, Y, t)}^{(S_s, Y, t)} r(s, y, v) dv] [r(S_s, Y, s)K - \delta(S_s, Y, s)S_s] \) in the exercise region. Under Assumption M (i) this is a decreasing functional of the trajectories of the asset price (i.e., \( \Phi \) becomes more negative) in the exercise region and (ii) trajectories of the asset price are nondecreasing (\( P - a.s. \)). Thus, delaying exercise at \((S, Y, t)\) becomes more costly if the initial asset price \( S \) increases and \((S, Y, t) \in \mathcal{E}\).

The result of Proposition 2 implies that the American call option satisfies the EEP Formula (3)–(4), substituting \( B(Y, t) \) for \( B(t) \). Similarly, the immediate exercise boundary surface is a multivariate function of \((Y, t)\), which solves (5).

Having established the results of Propositions 1 and 2, we now turn to applications.

3. American Options with Stochastic Volatility

In this section we value American options when the underlying asset price exhibits stochastic volatility. Specifically, based on the characterization of Proposition 1, we provide analytic formulas for the exercise boundary and the option price when the asset price follows a CEV process. These formulas will be used later in our study of capped options.

It is well known that the implied volatility of market indices often exhibits a smile/skew structure. One popular approach to model this feature of the data is to specify a diffusion process which incorporates the negative correlation between changes in the asset price and the volatility. A process which displays this property is the constant elasticity of variance process (CEV) introduced by Cox and Ross (1976),

\[
dS_t = (r - \delta)S_t dt + \sigma S_t^{\theta/2} dW_t,
\]

with \( r, \delta, \sigma > 0 \) and where \( r, \delta, \sigma \) are constants.

To evaluate derivatives written on this process we need the density function of the price under the risk-neutral measure. If \( \theta = 2 \), this is the log-normal density. For \( 0 < \theta < 2 \) the density is provided by Cox (1975, 1996), Emanuel and MacBeth (1982), or Schroder (1989).\(^{14}\) Computing the expectations in (3)–(4) based on this density function leads to a valuation formula proposed by Kim and Yu (1996). Their derivation assumes an exercise set of the form \( \mathcal{E} = \{(S, t) \in \mathbb{R}_+ \times [0, T] : S \geq B(t)\} \). Proposition 1 validates this conjecture.

**Proposition 3.** Consider an American call option with maturity \( T \) and strike \( K \). Its optimal exercise boundary \( B(t) \) satisfies the recursive integral Equation (5) with

\[
C'(B(t), t) = B(t)e^{-\delta(T)}\phi_1(B(t), K, T)
- Ke^{-r(T)}\phi_2(B(t), K, T),
\]

\[
\Pi(B(t), t; B(\cdot)) = \int_{t}^{T} (\delta B(t)e^{-\delta(v)}\phi_1(B(t), B(v), v)) \, dv,
\]

\( ^{14} \) Davydov and Linetsky (2000) describe the boundary behavior of the process.
where

\[
\phi_1(B(t), B(v), v) = \begin{cases} 
\chi^2(2y_B(v); 2 + \frac{2}{2 - \theta}, 2x_B(v)) & \text{if } \theta < 2 \\
\chi^2(2x_B(v); \frac{2}{\theta - 2}, 2y_B(v)) & \text{if } \theta > 2,
\end{cases}
\]

\[
\phi_2(B(t), B(v), v) = \begin{cases} 
1 - \chi^2(2x_B(v); \frac{2}{2 - \theta}, 2y_B(v)) & \text{if } \theta < 2 \\
1 - \chi^2(2y_B(v); 2 + \frac{2}{\theta - 2}, 2x_B(v)) & \text{if } \theta > 2,
\end{cases}
\]

and \( \tau(v) = v - t \). The function \( \chi^2(x; v, y) \) is the complementary noncentral chi-square distribution function evaluated at \( x \), with \( v \) degrees of freedom and noncentrality parameter \( y \). The integral equation is subject to the boundary condition \( B(T -) = \max(K, r/SK) \).

The integral equation for the exercise boundary is similar in structure to the lognormal case, except for the fact that \( \chi^2 \) distribution functions appear in the integrand instead of normal distributions. As a result, standard numerical methods used to solve the integral equation for the lognormal case can in principle also be used for the CEV model.

Next we illustrate properties of the formula. Implementation is based on the recursive algorithm suggested by the integral equation and described for a more general model in §7.1.\(^{15}\)\(^{16}\) The number of time steps in the algorithm is set at \( n = 2,000 \).

Figure 1 graphs the exercise boundary for values of \( \theta \) equal to 1, 1.3, 1.6, and 1.9 (in ascending order). The interest rate is \( r = 6\% \), the dividend rate \( \delta = 4\% \), and the option strike \( K = 100 \). Approximation based on the integral equation with 2,000 time steps.

\[B = \begin{array}{c}
150 \\
160 \\
170 \\
180 \\
190 \\
200 \\
210 \\
220 \\
230 \\
240 \\
\end{array}
\]

\[T-t\]

\[0.1 \quad 0.2 \quad 0.3 \quad 0.4 \quad 0.5 \quad 0.6 \quad 0.7 \quad 0.8 \quad 0.9 \quad 1\]

Note. The four curves correspond to elasticities of variance \( \theta = 1, 1.3, 1.6, \) and 1.9 (in ascending order). The interest rate is \( r = 6\% \), the dividend rate \( \delta = 4\% \), and the option strike \( K = 100 \). Approximation based on the integral equation with 2,000 time steps.

Figure 1 displays the corresponding results for the American option price. As anticipated, the price increases at all moneyness ratios when \( \theta \) increases. Across elasticities, the American option price exhibits the standard increasing behavior with respect to the underlying asset price. Figure 3 illustrates an

\[\begin{array}{c}
\chi^2(x; v, y) = \frac{2(r - \delta)}{\sigma^2(2 - \theta)(e^{(r - \delta)(2 - \theta)\tau(v)} - 1)} \\
x_B(v) = \kappa(v)B^2(2 - \theta)(2 - \theta)\tau(v) \\
y_B(v) = \kappa(v)B^2(2 - \theta).
\end{array}\]
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Figure 2 CEV Model: American Call Price (y Axis) Versus Asset Price (x Axis)

Note. The three curves are for elasticities of variance $\theta = 1, 1.6, 1.9$ (in ascending order). The interest rate is $r = 6\%$, the dividend rate $\delta = 4\%$, the option's maturity $T = 1$ year, and its strike price $K = 100$. Approximation based on the integral equation with 2,000 time steps.

An interesting property of CEV models, which is implied by the fact that the EEP does not increase at the same rate across elasticities. This is a crossing property of the premia associated with different values of $\theta$. At low prices the EEP associated with a larger value of $\theta$ is greater because the underlying price is more volatile. At intermediate prices the reverse holds. This follows because the American option value associated with a larger $\theta$ increases more slowly in between the corresponding exercise boundaries. Eventually, both premia converge to the same value as the probability of the European options being out of the money becomes null.

Figure 3 CEV Model: Early Exercise Premium (y Axis) Versus Asset Price (x Axis)

Note. The two curves are for elasticities of variance $\theta = 1$ and 1.9. The interest rate is $r = 6\%$, the dividend rate $\delta = 4\%$, the option's maturity $T = 1$ year, and its strike price $K = 100$. Approximation based on the integral equation with 2,000 time steps.

4. American Bond Options in One-Factor Models

In the fixed-income market, many securities can be viewed as American-style bond options. For instance, Bermudan and American swaptions are written on swaps, which are coupon-bearing bonds. Mortgages are fixed-income securities with an American-style option to repay the mortgage back to the lender at its face value.

As discussed in §2.1, the EEP representation holds even when the interest rate is stochastic and calibrated to the initial yield curve. We now examine an application of this result to the valuation of American bond options. The general formulas derived below will then be refined in a parametric model with quadratic term structure.

Suppose that the interest rate $r$ follows the (risk-neutralized) diffusion process $dr = \mu(r, t) dt + \sigma(r, t) dW$, where the functions $\mu(r, t)$ and $\sigma(r, t)$ satisfy suitable conditions. We seek to value an American call bond option with strike $K$, maturity date $T$, written on a coupon-bearing bond which matures at $T^* > T$. Let $P(r, t, s)$ denote the price at time $t$, in state $r$, of the zero-coupon bond maturing at time $s \geq t$. If the underlying coupon-bearing bond has continuous deterministic coupon payment $D$, its date $t$-value, $P_D(r, t, T^*)$, is

$$P_D(r, t, T^*) = P(r, t, T^*) + \int_t^{T^*} D(r, t, s) ds. \quad (15)$$

The main difficulty in valuing the American bond option stems from the stochastic nature of the interest rate. Because in a one-factor model there is a one-to-one relation between the interest rate and the coupon-bearing bond price, this bond price can be taken as
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Furthermore, note that the interest rate \( r(P_D(t, T_*)) \) is a decreasing function of the bond price and that the coupon on the bond \( D \) is independent of the bond price. As a consequence, the results of Proposition 1 can be applied to value the option. The resulting formula was obtained by Jamshidian (1992), who assumed that it is optimal to exercise when the interest rate falls below some critical interest rate (equivalently, the bond price exceeds a threshold). This is established by our Proposition 1 above.

**PROPOSITION 4.** There exists an exercise boundary curve \( B_D(t) \), \( t \in [0, T] \) such that the American contract is exercised at time \( t \) if and only if \( P_D(t, T_*) \geq B_D(t) \).

Moreover, the EEP representation implies that the time \( t \) value \( C(P_D(t, T_*), t; B_D(\cdot)) \) of this contract is the sum of the European contract price \( C^*(P_D(t, T_*), t) \) and the early exercise premium

\[
\Pi(P_D(t, T_*), t; B_D(\cdot)) = \int_t^T E_t^d \left[ \exp \left( -\int_t^s r(P_D(u, T_*)) \, du \right) \left( D_s - r(P_D(s, T_*)) K \right) 1_{[P_D(s, T_*) \geq B_D(s) ]} \right] ds. \tag{16}
\]

The immediate exercise boundary \( B_D \) is a function of time which solves the recursive integral equation

\[
B_D(t) - K = C^*(B_D(t), t) + \Pi(B_D(t), t; B_D(\cdot)) \quad \text{subject to the boundary condition} \quad B_D(T_-) = \max\{K, P_D(D_T/K, T, T_*), T_*) \}.
\]

Alternatively, it is possible to express the exercise region and the valuation formula entirely in terms of the underlying interest rate. This is again rationalized by the one-to-one relation between the bond price and the interest rate in a one-factor model.

**PROPOSITION 5.** There exists a critical interest rate \( r^*_i \) such that the exercise region is given by \( r_i \leq r^*_i \), where \( r^*_i \) is uniquely determined by \( P_D(r^*_i, t, T) = B_D(t) \).

As the proposition shows, immediate exercise is optimal at date \( t \) if and only if \( r_i \leq r^*_i \), i.e., if and only if the interest rate reaches a lower barrier. This structure of the exercise set is a natural consequence of the fact that the bond price (the underlying asset) is a strictly decreasing function of the interest rate for the class of one-factor models under consideration.

(see Figure 4 for an illustration in the quadratic term structure model).

Using the critical interest rate curve \( r^*_i \) enables us to write yet another expression for the EEP. For this purpose, use the bond price \( P(r, t, s) \) as the numeraire to write (see Jamshidian 1992, Equation (20)):

\[
\Pi(\cdot) = \int_t^T P(r, t, s) E_t^d \left[ (D_s - r_s K) 1_{[D_s \geq B_D(s) ]} \right] ds,
\]

where \( E_t^d \) is the expectation under the new measure \( Q^d \). Under this measure, called the adjusted-forward \( s \)-measure, the spot rate process is

\[
\frac{dr}{r} = \left[ \mu(r, u) + \sigma(r, u) \left( \frac{\partial}{\partial r} \ln P(r, u, s) \right) \right] du + \sigma(r, u) dW^*_s,
\]

and \( W^* \) is a \( Q^d \)-Brownian motion process (see proof of Proposition 5 in the technical appendix). Jamshidian (1992) obtained an explicit integral representation of the EEP in one-factor Vasicek and one-factor CIR models; Chesney et al. (1993) also give the representation for the CIR model. We now provide a new example in which the representation is explicit, namely, the case of the quadratic term structure model.
4.1. Bond Options in the Quadratic Term Structure Model

We first briefly review key elements of the quadratic model of the term structure. Suppose that the state variable $y$ follows the diffusion process $dy_t = a(y_t - y^*) dt + \sigma dW_t$, and the instantaneous spot rate $r_t = y_t$. The dynamics of $r$ is then given by $dr_t = [2ay_t(y_t - y^*)] dt + 2\sigma y_t dW_t$. This term structure model is called "quadratic" because of the exponential quadratic formula for a zero-coupon bond price,$^{17}$

$$P(r, t, T) = \exp \left( -A_1(t, T)y_t^2 - A_2(t, T)y_t + A_3(t, T) \right),$$

(17)

where

$$A_1(t, T) = \frac{1 - \exp[-2\gamma(T - t)]}{[\gamma + a] + [\gamma - a] \exp[-2\gamma(T - t)]},$$

(18)

$$A_2(t, T) = \int_t^T n(u) \exp \left( -\int_u^t m(s) ds \right) du,$$

(19)

$$A_3(t, T) = \int_t^T \left[ -a\kappa A_2(s, T) + \frac{1}{2} A_2(s, T)\sigma^2 - A_1(s, T)\sigma^2 \right] ds,$$

(20)

with $\gamma = \sqrt{a^2 + 2\sigma^2}$, $m(s) = 2\sigma^2 A_1(s, T) + a$ and $n(u) = 2a\kappa A_1(u, T)$.

A nice feature of this model is that the short rate is positive. Besides, it was shown in Boyle and Tian (1999) that the process $y_t$ has a normal distribution under the T-forward risk-adjusted measure. It follows that analytic valuation formulas for European bond options and caps/floors can be derived. Our next proposition gives a simple expression for the early exercise premium which in turn leads to a recursive integral equation for the critical interest rate $r^*_t$.

**Proposition 6.** Consider an American call option on a coupon-bearing bond, in the quadratic term structure model. The option price $C(r_t, t; r^*_t)$ satisfies the EEP formula where the European bond option price $C^*(r_t, t)$ is given in the technical appendix and the exercise premium equals

$$\Pi(r_t, t; r^*_t) = \int_t^T P(r_t, t, s)Q(s) ds,$$

(21)

where

$$Q(s) = (D_s - K(M(t, s, r_t)^2 + \Sigma_s))(N(z_{1s}) - N(z_{2s})) + 2\sqrt{\Sigma_s} KM(t, s, r_t)(n(z_{1s}) - n(z_{2s})) + K\Sigma_s(z_{1s} n(z_{1s}) - z_{2s} n(z_{2s})),$$

(22)

with

$$z_{1s} = \frac{\sqrt{r_t^2 - M(t, s, r_t)}}{\sqrt{\Sigma_s}}$$

and

$$z_{2s} = -\frac{\sqrt{r_t^2 - M(t, s, r_t)}}{\sqrt{\Sigma_s}},$$

(23)

$$M(t, s, r_t) = \sqrt{r_t} \exp \left( -\int_t^s \alpha(v) dv \right) + \int_t^s \exp \left( -\int_v^s \alpha(u) du \right) \beta(v) dv,$$

(24)

$$\Sigma_s = \sigma^2 \int_t^s \exp \left( -\int_u^s \sigma^2 (v) dv \right) du,$$

(25)

and $\alpha(v) = a + 2A_1(v, T)\sigma^2$ and $\beta(v) = ak - A_2(v, T)\sigma^2$.

The immediate exercise boundary solves the recursive integral equation $P(r^*_t, t, T^*_t) - K = C^*(r^*_t, t) + \Pi(r^*_t, t; r^*_t)$ subject to the boundary condition $P(r^*_T, T, T^*_T) = K$.

Note that the integral equation which characterizes the exercise boundary $r^*$ is obtained in explicit form. It follows that the procedure employed in earlier sections can also be used in implementations of the bond-option pricing formula. The illustration in Figure 4 shows that the exercise boundary $r^*$ is increasing in time (i.e., it increases as maturity approaches). This reflects the inverse relationship between the bond price and the interest rate. When converted to a threshold level for the bond price the (price) boundary is decreasing in time.

5. American Options with Stochastic Interest Rates

We now examine the valuation of American options when interest rates are stochastic. Interest rate risk is particularly relevant for long-dated options. Long-term derivatives are quite common in the exchange and OTC markets. For example, equity options, swaps, and LEAPS all trade with maturities longer than two years. Commodity and energy options trade with
even longer maturities, of five and ten years. Interest rate risk is then a significant factor in the valuation of these contracts. Moreover, because short-term rates are imperfectly correlated with stock returns, a multivariate model is needed. As illustrated below, the results of §2.1 might apply. If this is the case, by Proposition 2 the exercise region is determined by a unique exercise surface, and at each date the exercise region involves a single critical curve which depends on the short rate. This property, which is proved here for the first time, has been standard practice and is consistent with previous observations made by other authors (e.g., Ho et al. 1997). Moreover, the property can be used to characterize the immediate exercise boundary.

Suppose that the underlying asset price $S$ and the interest rate $r$ follow the bivariate risk-neutralized process

$$
\frac{dS_t}{S_t} = (r_t - \delta) dt + \sigma_1 dW_{1t},
$$

$$
dr_t = [\theta(t) - ar_t] dt + \sigma_2 dW_{2t},
$$

where $(\delta, a, \sigma_1, \sigma_2)$ are constants, $\theta(t)$ is deterministic, and $W_1, W_2$ are correlated Brownian motions (with correlation coefficient $\rho$). The coefficient $\delta$ is the dividend rate, $\sigma_1$ the asset price volatility, $a$ the speed of mean reversion of the interest rate, and $\sigma_2$ its volatility.

Previous authors (Amin and Bodurtha 1995, Ho et al. 1997, Chung 1999, Menkveld and Vorst 2000) have considered the model (26)-(27) for $(S, r)$, but with null dividend ($\delta = 0$). In this context, the adjusted-forward $T$-measure was introduced and used to derive valuation formulas for European puts and Bermudan options with two exercise dates. Combining these formulas with a two-points Richardson extrapolation scheme they were then able to approximate the values of American options. This approach to the valuation of American options is fast, but does not provide any information about the optimal exercise policy. Moreover, the approximation obtained is coarse. In principle, accurate valuation could be obtained by increasing the number of exercise dates, but this would require explicit (or computationally tractable) formulas for Bermudan options with a sufficiently large number of exercise dates. As shown below, the EEP representation provides an alternative approach which may be more transparent to the extent that it provides detailed information about the structure of the American option price and its exercise boundary. 18

In the market (26)-(27) the pure discount bond price is $P(t, T) = J(t, T) \exp(-r_t G(t, T))$, where

$$
J(t, T) = \exp\left(-\int_t^T \int_t^v e^{-s(t-u)} \theta(s) ds dv \right.

\left. + \frac{1}{2} \sigma_2^2 \int_t^T (1 - e^{-a(T-u)})^2 ds \right),
$$

$$
G(t, T) = \frac{1}{a} \left[1 - e^{-a(T-t)}\right].
$$

The European call price is given by (see technical appendix):

$$
C^*(S_t, r_t, t) = e^{-\delta(T-t)} S_t N(h(S_t, K; t, T)) - KP(t, T)
\times N(h(S_t, K; t, T) - \sqrt{w(t, T)}),
$$

with

$$
h(S, K; t, T) = \frac{\ln(S/KP(t, T)) - \delta(T-t) - \frac{1}{2} \sqrt{w(t, T)}}{\sqrt{w(t, T)}},
$$

$$
w(t, T) = \int_t^T (\sigma_1^2 + \sigma_2^2 G(u, T)^2 + 2\rho \sigma_1 \sigma_2 G(u, T)) du.
$$

To apply the results of Proposition 2 in the current context we must verify that the requirements of Assumption M hold. Clearly, condition (iii) is satisfied because the dividend yield $\delta$ is a constant. Similarly, (ii) is verified since the coefficients of the interest rate process do not depend on the asset price. Finally, condition (i) holds for the same reason: The coefficients of (26) do not depend on the asset price.

Using the results of Proposition 2 enables us to state:

**Proposition 7.** Consider an American call option with maturity $T$ and strike $K$ in the financial market (26)-(27).

A recombining two-dimensional tree or a finite difference method could also be used to get accurate results. See Wiggins (1987) for the application of a PDE method to price options in a two-dimensional problem with stochastic volatility.
The exercise region is $\mathcal{C} = \{(S, r, t) \in \mathbb{R}_+ \times \mathbb{R} \times [0, T] : S \geq B(r, t)\}$ for a two-variable function $B(r, t)$. Moreover, $B(r, t)$ satisfies the recursive integral equation $B(r, t) - K = C'(B(r, t), r, t) + \Pi(B(r, t), r, t; B(\cdot, \cdot))$ subject to the boundary condition $\lim_{t \to T} B(r, t) = \max\{K, rT/\delta K\}$. The European option value $C^e$ is defined in (30)-(32). The exercise premium $\Pi$ is given by

$$\Pi(S, r, t; B(\cdot, \cdot)) = \delta S \int_t^T P(t, v) e^{-\delta(v-t)} \phi_1(S, r, t; B(\cdot, v), v) dv - K \int_t^T P(t, v) \phi_2(S, r, t; B(\cdot, v), v) dv,$$

where $P(t, T)$ is the bond price and

$$\phi_1(S, r, t; B(\cdot, v), v) = \frac{1}{\sigma^t(v)} \exp\left(\frac{1}{2} \tilde{\sigma}^2(t, v)\right) \int_{-\infty}^\infty \exp(\tilde{\mu}(r, t, v)) N(g(r; S, B(r, v), t, v)) n\left(\frac{r - \mu^t(r, t, v)}{\sigma^t(v)}\right) dr,$$

$$\phi_2(S, r, t; B(\cdot, v), v) = \frac{1}{\sigma^t(v)} \int_{-\infty}^\infty r N(g(r; S, B(r, v), t, v) - \tilde{\sigma}(t, v)) \times n\left(\frac{r - \mu^t(r, t, v)}{\sigma^t(v)}\right) dr.$$

The functions appearing in $\phi_1, \phi_2$ are

$$g(r, B(r, v); S, r, t, v) = \ln(S/B(r, v)) + \frac{\tilde{\mu}(r, t, v) - \delta(v-t) + \tilde{\sigma}(t, v)^2}{\tilde{\sigma}(t, v)},$$

$$\tilde{\mu}(r, t, v) = \mu^*(r, t, v) + \frac{\rho \sigma^*(t, v) \sigma^t(v)}{\sigma^t(v)},$$

$$\tilde{\sigma}^2(t, v) = \sigma^*(t, v)^2 (1 - \rho \sigma^*(t, v)^2),$$

$$\mu^*(r, t, v) = r e^{-\delta(v-t)} + \int_t^v e^{-\delta(v-s)} (\theta(s) - \sigma^2(s, v)) ds,$$

and

$$\sigma^*(t, v)^2 = \sigma^2 \int_t^v e^{-2\delta(v-s)} ds,$$

with

$$\mu^*(r, t, v) = r(t) - \frac{1}{2} \sigma^2(t)(v-t) + \int_t^u \int_t^v e^{-\delta(s-u)} \theta(u) dv du - \int_t^v e^{-\delta(s-u)} \sigma^2(s, v) du,$$

and $G(t, v)$ defined in (29).

To implement the formulas of Proposition 7 we follow the scheme outlined in $\S 7.1$. Identification of the exercise region in a subset $\mathbb{R} \times [0, T]$ where $\mathbb{R}$ is performed by truncating the domain of the interest rate to an interval $[r, \bar{r}]$, which contains $\mathbb{R}$. This interval $[r, \bar{r}]$ is chosen to be sufficiently large relative to $\mathbb{R}$ so as to limit the (truncation) bias in the subset of interest.

Figure 5 displays the typical shape of the boundary surface when $\theta$ is constant. As expected, the exercise boundary is nonincreasing in time to maturity. It is also nonincreasing in the interest rate. This reflects the reduction in the benefits from early exercise associated with an increase in the opportunity cost. In the limit, as maturity approaches the boundary surface converges to the piecewise linear function of the interest rate $B(r, T_-) = \max\{1, r/\delta K\}$.

6. American Capped Call Options

A large number of securities that have been issued by financial institutions or are exchange traded include cap features combined with standard option payoff structures. These contracts are especially attractive for issuers because they imply limited liability. At the same time, they can still provide the incentive of an upside potential to investors.

In this section we consider capped call options where the cap $L(t)$ satisfies $dL(t) = g(t)L(t) dt$, with
The option strike is \( K = 100 \) and the cap \( L = 210 \). The interest rate is \( r = 6\% \), the dividend rate \( \delta = 3\% \), the volatility coefficient \( \sigma = 0.3 \), and the elasticity of variance \( \theta = 1.8 \). Approximation based on the integral equation with 2,000 time steps.

Note. The call strike is \( K = 100 \). The asset pays dividends at the rate \( S = 0.05 \) and its return volatility is \( \sigma_0 = 0.2 \). The interest rate process has long-term mean \( \bar{r} = \delta/a = 0.06 \), speed of mean-reversion \( a = 0.005 \), and volatility \( \sigma_f = 0.01 \). The interest rate and the asset are uncorrelated, \( \rho = 0 \). Approximation based on the integral equation with a grid of 100 x 50 points for \((T, r)\).

Underlying asset price follows the general diffusion process (1) with decreasing interest rate and increasing dividend. The immediate exercise region of the capped call option is \( \mathcal{E}(L) = \{(S, t) \in \mathbb{R}^+ \times [0, T] : S \geq B(t) \wedge L\} \), where \( B(t) \) is the exercise boundary of the corresponding uncapped option with identical characteristics \((T, K)\).

This result illustrates a remarkable connection between capped and uncapped options, written on the general diffusion process (1) because it states that the capped option exercise boundary is the minimum of the boundary for the uncapped option and the cap (see Figure 6 for an illustration in the CEV model). As a result, the boundary for the capped option is completely identified once the exercise boundary for the uncapped option is known.

This connection was initially shown for the case of lognormal underlying asset price process by Broadie and Detemple (1995). As our proposition demonstrates, it is also valid in much more general contexts where the interest rate, the dividend rate, and the volatility of the asset depend on the price \( S \). The intuition for the generality of the result is as follows. Clearly, waiting to exercise when \( S \) is above the cap is suboptimal due to the time value of money. When

\[ L(0) > K \] and \( g(t) \geq 0 \). Here \( g(t) \) represents the growth rate of the cap, which is a time-dependent function. The subcases of constant cap \( L(t) = L (g = 0) \) and cap with a constant growth rate \( g(t) = g \) were considered in Broadie and Detemple (1995) under the additional assumption of a lognormal underlying asset price.

The American capped call option payoff is \((S \wedge L(t) - K)^+)\). Let \( C^L(S, t) \) denote its price. Also let \( \mathcal{E}(L) = \{(S, t) \in \mathbb{R}^+ \times [0, T] : C^L(S, t) = (S \wedge L(t) - K)^+\} \) be the immediate exercise region. While the original deductions in Broadie and Detemple (1995) depend heavily on the (lognormal) distribution of the underlying asset prices, our arguments below are more conceptual and work for general diffusion processes (1). In addition to the previous assumptions, we also suppose that the interest rate is positive.

### 6.1. American Options with Constant Caps

We first focus on the case of a constant cap \( L \) and characterize the exercise set in terms of the exercise boundary of the corresponding uncapped option.

**Proposition 8.** Consider an American capped call option with constant cap \( L(t) = L \) and suppose that the underly-
S is below the cap and \( B(t) \leq L \), the holder of the capped option can implement the exercise policy of the corresponding uncapped option (see Figure 6). This policy will thus be optimal for the capped option as well. From these two observations we conclude that the exercise boundary for the capped option \( B^c \) is always bounded above by \( B \wedge L \) and equal to \( B \) at the first time \( t_* \) at which \( B(t) = L \). Moreover, a straightforward extension of Proposition 1 shows that \( B^c \) is nonincreasing. Combining these elements shows that \( B^c = B \wedge L \).

An alternative explanation for the suboptimality of exercise when \( S < B \wedge L \) can also be provided along the lines of Broadie and Detemple (1995). Suppose that we stand at time \( t \) and that \( S < L < B \). In this event the holder of the capped option can always find an uncapped option with the same strike, but a shorter maturity date \( T_0 \leq T \), whose exercise boundary \( B(t, T_0) \) satisfies \( S < B(t, T_0) < L \) at time \( t \) and \( B(v, T_0) < L \) at all times \( v \in [t, T_0] \). Moreover, following this exercise policy produces the payoff of the uncapped option with maturity \( T_0 \) (since \( B(v, T_0) < L \) for all \( v \in [t, T_0] \)). We conclude that \( C(t, T_0) \geq C(t, T) \), and since \( C(t, T) > S - K \), it follows that immediate exercise is also suboptimal for the holder of the capped option at date \( t \). In essence, the capped option holder has been able to find a waiting policy which dominates immediate exercise.

The relation between the two exercise boundaries has several advantages. One of these is that it simplifies the computation of American capped call option values. Indeed, once the boundary of the uncapped option is known there is no need to recompute a boundary for the capped option. It also follows from this feature that the capped option price can be calculated using any standard numerical method.

6.2. American Options with Time-Dependent Caps

Let us now consider the case of time-dependent cap \( dL(t) = g(t)L(t) \, dt \), with \( L(0) > K \) and \( g(t) \geq 0 \). To simplify the analysis we assume (i) that the interest rate is a constant \( r \), and (ii) that the discounted cap-payoff function \( \Psi(t) = e^{-rT}(L(t) - K) \) has only one (essentially) maximal point at \( T = t_* \). Condition (i) could be relaxed because any deterministic interest rate will work. Condition (ii) is automatically satisfied when the growth rate \( g(t) = g > 0 \) is a positive constant, which is the case studied in Broadie and Detemple (1995).

To identify the exercise region we consider the following class of exercise policies called \( (t_*, t^*, t_f) \)-policy (see Figure 7 for an illustration in the CEV model). Recall that \( B(t) \) denotes the exercise boundary for the uncapped option. Let \( t_* \) and \( t_f \) satisfy \( 0 \leq t_* \leq t_f \leq T \) and \( t_* \leq t^* \leq T \), where \( t^* \) is the minimal solution of the equation \( B(v) = L(v) \) for \( v \in [0, T] \), if such a solution exists. If \( B(v) < L(v) \) for all \( v \in [0, T] \), set \( t^* = 0 \), and if \( B(v) > L(v) \) for all \( v \in [0, T] \), set \( t^* = T \). If the cap curve \( L(v) \) does not meet the boundary curve \( B(v) \) because of a possible discontinuity in \( B \); i.e., if \( L(v) \) cuts \( B(v) \) at some unique (because \( B \) is decreasing while \( L \) is increasing) point of discontinuity, set \( t^* \) equal to this point of discontinuity. Now define the stopping time \( \tau_1 \) by

\[
\tau_1 = \inf\{ v \in [t_*, t_f] : S(v) = L(v) \} \tag{42}
\]
or, if no such \( v \) exists, set \( \tau_1 = T \). Set the stopping time \( \tau_2 \) equal to \( \tau_2 = t_f \) if \( S(t_f) \leq L(t_f) \); otherwise set \( \tau_2 = T \).
Define the stopping time $\tau_3$ by

$$\tau_3 = \inf\{v \in [t^*, T] : S(v) = B(v)\} \quad (43)$$

or, if no such $v$ exists, set $\tau_3 = T$. An exercise policy is a $(t_e, t^*, t_f)$-policy if the option is exercised at the stopping time $\tau = \min\{\tau_1, \tau_2, \tau_3\}$.

With this definition we can state the following result,

**Proposition 9.** Suppose that $r$ is constant and that the discounted cap payoff $\Psi(t)$ has a unique maximum. Then the optimal exercise policy is a $(t_e, t^*, t_f)$-policy, where $t^*, t_f$ are defined above and $t_e$ is determined by solving a univariate optimization problem.

This proposition establishes a robustness property of the result achieved in Broadie and Detemple (1995) for constant growth rates and lognormal underlying asset price. As stated above, the optimality of the class of $(t_e, t^*, t_f)$-policies is impervious to the structure of the dividend rate and the volatility of the underlying asset price. This is of considerable interest because even for complex diffusion environments the boundary is now known to be parameterized by three constants. Moreover, one of these parameters, $t_f$, depends only on the structure of the discounted cap function $\Psi(t)$ and can be computed using a simple univariate optimization scheme. The second parameter, $t^*$, is also completely identified once the exercise boundary for the corresponding uncapped option has been computed. Finally, the last parameter, $t_e$, is obtained by maximizing the value of the $(t_e, t^*, t_f)$-policy relative to $t_e$. As mentioned above, this is a univariate optimization problem which can be resolved using standard methods.

Intuition for the optimality of a $(t_e, t^*, t_f)$-policy is as follows. First, note that immediate exercise is optimal for the capped option in the region $B < S < L$ because it is optimal for the uncapped option and the payoffs are the same. Similarly, when $t \geq t^*$ and $S < B(t)$, the capped-option holder can implement the exercise policy of the uncapped option and collect the same payoff. Second, note that immediate exercise is also optimal if $t \geq t_f$ and $S > L(t)$. In this region the discounted payoff $e^{rt}(L(t) - K)$ decreases almost surely and, as a result, any waiting policy reduces value. The reverse argument applies when $t < t_f$ and $S > L(t)$. Here, the discounted payoff increases over time, which means that the policy of exercising at $t_f \wedge \tau_L$, where $\tau_L$ is the first hitting time of the cap $L$, increases value. Third, when $t < t^*$ and $S < L(t)$, one can find an uncapped option with shorter maturity whose exercise boundary falls below the cap at all subsequent times. Implementation of this exercise policy dominates immediate exercise for the capped-option holder. Lastly, along the cap (i.e., when $S = L$ and $t < t^*$) the benefits of waiting (which dominate above the cap) are curtailed by the risk of falling below the cap. When the maturity date is sufficiently close (i.e., for $t$ close to $t^*$), this risk is enough to prompt immediate exercise. With longer horizons the appreciation in the discounted value of the cap is sufficient for the optimality of a waiting policy.

### 7. Numerical Methods

Various numerical schemes have been proposed to compute the exercise boundary when the underlying asset price is a geometric Brownian motion (see Broadie and Detemple 1996 for a review). In principle, each of these could be adapted to the case of general diffusion process. In this section we focus more specifically on two approaches that have received attention in the recent literature, namely integral equation methods (§7.1) and stopping-time approximations (§7.2).

#### 7.1. Integral Equation Methods

To show that the approach is viable in a two-dimensional setting, we consider the model of §2.1 with a state variable $Y$. The numerical implementation procedure suggested by the EEP formula involves two steps. The first step consists of computing the optimal exercise boundary $B(Y, t)$, whose existence follows from Proposition 2. The second step is to compute the option price, taking the curve $B(Y, t)$ as an input.

Suppose that the truncated moments appearing in the integral equation can be written explicitly in terms

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19 This holds if $L(t) \geq (r/\delta)K$. If $L(t) \leq (r/\delta)K$, immediate exercise is suboptimal because the local benefit of exercising is negative.
of known density and cumulative distribution functions (as in the model of §5). For a call option, the integral equation can then be expressed as

\[ B(Y_i, t) - K \]
\[ = C'(B(Y_i, t), Y_i, t) + \int_{Y}^{\bar{Y}} \int_{t} \phi(B(Y, t), Y; B(Y, v), Y, v) dY dv, \]

where the functions \( C'(B(Y_i, t), Y_i, t) \) and \( \phi(B(Y_i, t), Y_i; B(Y, v), Y, v) \) involve the relevant distributions and \([Y, \bar{Y}]\) is the range of \( Y \).

The boundary can be computed as follows. Divide the period \([0, T]\) into \( n \) equal subintervals (0 stands for the current time) and let \( \Delta t = T/n \). Similarly, divide the range of \( Y \), i.e., the interval \([Y, \bar{Y}]\), into \( m \) equal subintervals and let \( \Delta Y = (\bar{Y} - Y)/m \). For \( i = 0, \ldots, m \) set \( Y(i) = Y + i\Delta Y \). For \( j = 0, \ldots, n \) set \( t(j) = j\Delta t \). We seek a step function approximation \( B(i, j) = B(Y(i), t(j)) \), for \( i = 0, \ldots, m \) and \( j = 0, \ldots, n - 1 \), which can be computed recursively.

Note that \( B(i, n) = \max(K, (r/\delta)K) \) for all \( i \). Suppose \( B(i, l) \) is known for all \( i \) and \( l > j \). A nonlinear equation for \( B(i, j) \) is obtained from the early exercise premium representation, as follows (\( i = 0, \ldots, m - 1 \) and \( j = 0, \ldots, n - 1 \))

\[ B(i, j) - K \]
\[ = C'(B(i, j), Y(i), t(j)) + \phi(B(i, j), Y(i); B(i, j), Y(i), t(j)) \Delta Y \Delta t \]
\[ + \sum_{p=0}^{m} \sum_{q=1}^{n-j-1} \phi(B(i, j), Y(i)); B(i+p, j+q), Y(p), t(j+q)) \Delta Y \Delta t. \]

Thus, in each state \((i, j)\) the exercise boundary approximation \( B(i, j) \) can be solved for in terms of the known future values \([B(i, k), i = 0, \ldots, m, k = i + 1, \ldots, n]\). This computation can be performed using standard iterative procedures.

The algorithm described above produces an approximate exercise boundary, hence an approximate option value, for any fixed number of time steps \( n \). The true price is the limit as \( n \) goes to infinity. To improve efficiency, one may use a standard Richardson extrapolation scheme \( C(S, t) = 2C(n) - C(n - 1) \), where \( C(n) \) denotes the current price with \( n \) steps. Alternative extrapolation schemes adapted from the one proposed by Geske and Johnson (1984), \( C(S, t) = 4.5C(n) - 4C(n - 1) + 0.5C(n - 2) \) could also be employed.

7.2. Stopping Time Approximations: Lower and Upper Bound Method (LBA, LUBA)

This procedure is based on an approximation of the optimal exercise policy obtained by computing the best policy among a collection of simple (suboptimal) exercise policies (see Broadie and Detemple 1996). The approach produces (i) a lower bound for the exercise boundary, (ii) lower and upper bounds for the option price, and (iii) approximate option values (LBA and LUBA).

The results to be described below are valid for general diffusions (1) even when the relevant moments in the EEP formula do not have an explicit form.

Formally, the procedure is based on the derivative of an American capped call option with respect to the cap \( L \). Write \( C(S, t, L) \) for the value of a capped call with automatic exercise at the cap \( L \), which has the same strike \( K \) and maturity date \( T \) as the American call option to be priced. Because the policy of exercising at the first time at which the asset price reaches the level \( L \) is feasible for the American call, the lower bound \( C(S, t) \geq C(S, t, L) \) for all \( L > 0 \) holds. Maximizing over \( L \) yields the highest lower bound in the class of suboptimal policies under consideration, \( C(S, t) \geq \max_{L>0} C(S, t, L) \). Now define

\[ D(L, t) = \lim_{S \downarrow L} \frac{\partial C(S, t, L)}{\partial L} \]  

(44)

With one-dimensional lognormal processes step function approximations were used by Huang et al. (1996). Alternative approximations by single-piece (e.g., Omberg 1987) or multipiece (e.g., Ju 1998) exponential functions have also been studied.
and let $L_t^*$ denote the solution of the equation $D(L, t) = 0$. The quantity $L_t^*$ represents the lowest asset price (at date $t$) at which immediate exercise is optimal if the option holder is restricted to policies of exercising at the first time at which the asset price reaches a given level $L$ (see Broadie and Detemple 1996 for additional intuition). The following relation ties the optimal exercise boundary to $L_t^*$:

**PROPOSITION 10.** Suppose that the underlying asset price satisfies (1) and that the assumptions of §2 hold. Let $B(t)$ be the optimal exercise boundary for the American call option, and $L_t^*$ the exercise boundary solving $D(L, t) = 0$. Then $L_t^* \leq B(t)$, for all $t \in [0, T]$.

Thus, $L_t^*$ is a lower bound for the optimal exercise boundary $B(t)$. Substituting this lower bound for $B(t)$ in the EEP representation (3) can be shown to produce an upper bound for the American option price $C(S, t)$. This enables us to conclude that all the elements of the LBA and LUBA algorithms extend to general diffusion processes satisfying the conditions of §2. This approach becomes particularly attractive when the capped call option $C(S, t, L)$ has a closed-form expression. In this instance computation of $L^*$ and of the price bounds is easy and fast to perform.

8. **Conclusions**

In this paper we have examined the valuation of American options when the underlying asset price follows a diffusion. General results on the geometry of the exercise set were established. Explicit valuation formulas were then derived for models with stochastic volatility and stochastic interest rate. Equity options as well as options on coupon-bearing bonds were studied. Optimal exercise policies and valuation formulas were also obtained for capped call options. Strikingly, we found that the optimal exercise boundary of a capped option has the same structural features for all diffusion models. This robustness property substantially simplifies the valuation of these instruments because it reduces the determination of the optimal policy to the identification of a few key parameters. In the case of a constant cap the exercise boundary is known once the boundary of the corresponding uncapped option has been identified.

Many of our results extend to more general contracts. For instance, parts of our analyses adapt easily to cover monotone and convex smooth payoffs. The exercise set of these claims is again limited by a single boundary curve which satisfies a recursive integral equation. When the distribution of the underlying price is available in closed form the integral equation is amenable to numerical analysis and can be used to construct approximations. Although more difficult to obtain than for piecewise linear contracts such as options, numerical results can still be derived in this instance to provide insights about valuation.

Finally, we should mention that the hardest problems, in dealing with American options, concern high-dimensional problems. While our approach shows that a class of low-dimensional problems involving nonlinear diffusions can be handled using the EEP approach, it does not resolve the more general multidimensional case. This is an important topic for future research in the field.23,24

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**Appendix**

**PROOF OF PROPOSITION 1.** We first prove the existence of a nonincreasing right-continuous function $B(t)$ such that the exercise region $\mathcal{Z}$ is the set of points $(S, t) \in \mathbb{R}_+ \times [0, T] : S \geq B(t))$. For this purpose the following lemma proves useful.

**LEMMA 1.** Consider a claim whose payoff $0 \leq u(S) \leq S$ satisfies the Conditions (i) $u(S)$ is increasing, and (ii) $u(S') - u(S) \leq S' - S$ for all $S' \geq S$. Let $V(S, t)$ denote the rational price of the contract. Then, $V(S', t) - V(S, t) \leq S' - S$ for all $S' \geq S$.

23 Monte Carlo simulation may be the approach of choice for addressing high dimensional problems. For a review of the method see Boyle et al. (1997).

24 A technical appendix to this paper is available as an electronic companion on the Management Science website at (mansci.pubs.informs.org).
Lemma 1 gives a bound on the variation in the price function for a given change in the underlying price. Note that the conditions on the payoff are standard. Condition (i), for instance, is satisfied by a given change in the underlying price. Note that the conditions on call payoffs. The case of puts is also covered by these conditions by put-call symmetry (see Schroeder 1999 or Detemple 2001).

Proof of Lemma 1. Let $\tau$ be the optimal stopping time at the point $(S_0, t)$, i.e. $V(S_0, t) = E^*[b_0(S_0)]$, where $b_0(S_0) = \exp(\int_0^t r(S_u) du)$. Since $\tau$ is suboptimal at $(S_0, t)$ we have $V(S_0, t) \geq E^*[b_0(S_0)]$ with $b_0(S_0) = \exp(\int_0^t r(S_u) du)$, and taking the difference $\Delta_t = V(S_0, t) - V(S_0, t)$ gives $\Delta_t \leq E^*[b_0(S_0)] - E^*[b_0(S_0)]$.

$$\Delta_t \leq E^*[b_0(S_0) - u(S_0)] + E^*[b_0(S_0) - u(S_0)]. \quad (A1)$$

Consider the stochastic process (under the risk neutral measure $Q$).

$$X_t = X_0 + \int_0^t X_u dW_u, \quad s \in [0, T]$$

for two initial values $X_0^1 = S_0$, $X_0^2 = S_0$. Since $X_t^1 \leq X_t^2$, $(Q - a.s.)$ the comparison theorem for solutions of stochastic differential equations (Karatzas-Shreve 1988, Proposition 2.18, p. 293), implies $Q[X_t^1 \leq X_t^2]$ for all stopping times $\tau \in \mathcal{F}_t$. Since $b_0(S_0) + \int_0^t b_0(S_t) dS_t$ is a $Q$-martingale, Doob's optional sampling theorem gives

$$S_t = E^*[b_0(S_t) + \int_0^t b_0(S_u) dS_u] \quad (A3)$$

Define $I = E^*[b_0(S_t) - u(S_t)]$. The results above and the assumption in the lemma that $u(x) - u(y) \leq x - y$, for all $x \geq y$ now yield

$$I \leq E^*[b_0(S_t) - u(S_t)]$$

$$= S_t - E^*[\int_0^T b_0(S_u) dS_u] - E^*[b_0(S_t)]$$

$$= S_t - E^*[b_0(S_t)] - E^*[\int_0^T b_0(S_u) dS_u] + E^*[b_0(S_t)] - E^*[b_0(S_t)]$$

$$= S_t - \Delta_t + E^*[\int_0^T b_0(S_u) dS_u].$$

The inequality on the first line follows from the assumption $u(x) - u(y) \leq x - y$. The second and fourth lines use (A1)-(A4). Other lines employ simple algebra. Substituting this inequality in (A1) and collecting terms gives the following upper bound for $\Delta_t$.

$$S_t - \Delta_t = S_t - E^*[\int_0^T b_0(S_u) dS_u] + E^*[\int_0^T b_0(S_u) dS_u]$$

$$- E^*[\int_0^T b_0(S_u) dS_u] - E^*[\int_0^T b_0(S_u) dS_u]. \quad (A5)$$

Since $S_t \geq S_t$, $\forall u \in [t, T]$ and $\tau(x, t)$ is decreasing relative to $x$, we have $r(S_t, u) \leq r(S_t, u)$. Hence $\exp(\int_0^t r(S_u) du) \geq \exp(\int_0^t r(S_u, u) du)$. Since $S_t \geq u(S_t), we obtain

$$E^*[b_0(S_t) - u(S_t)] \geq 0. \quad (A6)$$

On the other hand, since $\delta(S, v)S_t \geq \delta(S, v)S_t$ by assumption and $b_t \geq b_t$, we have

$$E^*[\int_0^t (b_0(S_t, v) v \Delta_t - b_0(S_t, v) v \Delta_t) du] \geq 0. \quad (A7)$$

Combining (A6) and (A7) with (A5) gives, $V(S_t, t) - V(S_t, t) \leq S_t - S_t$, which proves the lemma.

The first part of Proposition 1 now follows easily. First, note that the call-option payoff satisfies the two conditions of Lemma 1. The lemma then implies $C(S_t, t) \leq C(S_t, t) + S_t - S_t$, for $S_t \geq S_t$. Next, suppose that immediate exercise is optimal at $(S_t, t)$ but not at $(S_t, t)$. The optimality of exercise at $(S_t, t)$ combined with the inequality above, implies $C(S_t, t) \leq S_t - S_t = S_t - S_t$, which contradicts the suboptimality of exercise at $(S_t, t)$. We conclude that immediate exercise must also be optimal at $(S_t, t)$. Since this holds for any $(S_t, t) \in \mathcal{F}_{\tau_t}$ the exercise region is up-connected. The existence of a unique boundary curve follows. Standard arguments demonstrate that the boundary is nonincreasing. This follows from the time monotonicity of the process and the properties of the interest rate function. The non-increasing property of the boundary, together with the continuity of the price function, imply that the exercise boundary must be right-continuous.

We now prove the second part of Proposition 1. The proof is by contradiction. Suppose that the interest rate is deterministic and that $B$ is not left-continuous at time $t$. Choose $S \in (B(t), \inf(B(s), s < t))$. Then $(S_t, t)$ belongs to the exercise region and $\mathcal{F}(S_t, t) = r_t - S_t$ in distribution (see Jaillet et al. 1990, Theorems 3.1 and 3.2 with the assumption of deterministic of interest rate) where $\mathcal{F}$ is the well-known Black-Scholes/Merton differential operator

$$\mathcal{F} = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S} r(S, t) - \frac{\partial C}{\partial S} \delta(S, t) + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S^2 \sigma(S, t)^2 - r(S, t)C.$$
up-connected at the point $(S_i, t)$. Moreover, if $(A8)$ holds for any $(S_i, t) \in \mathcal{C}$, then the exercise region is up-connected. This extension of Proposition 1 provides a very general condition (condition $(A8)$) under which the exercise region is up-connecteded. Note that condition $(A8)$ does not place any a priori restrictions on the shapes of the interest rate function $r(S, t)$, the dividend flow function $\delta(S, t)S$ and the contingent claim payoff $u(S, t)$. Rather, the requirements are implicit in $(A8)$ and amount to a joint restriction on the first moment of the discounted payoff of the claim. The assumptions in $\S 2$ imply that $(A8)$ is verified.

**Proof of Proposition 2.** Let $r$ be the optimal stopping time at $(S_i, Y_i, t)$, i.e., $C(S, Y, t) = E^*[\phi; S,J]$. Suboptimality of $\tau$ at $(S, Y, t)$ then gives $C(S, Y, t) \geq E^*[\phi; S,J]$ and consequently, $C(S, Y, t) - C(S, Y, t) \leq E^*[\phi; S,J]$. It now suffices to prove that the right hand side is bounded above by $S - S_i$. By condition (i) of assumption M we know that $S_i > S_i$ and $Y_i = Y_i$ implies $S_i \geq S_i$ for any $v \geq t$. Thus, for any stopping time $\tau$ we must have $S_i \geq S_i$. Moreover, conditions (ii)-(iii) ensure that inequalities (A6)-(A7) hold. It follows that Lemma 1 applies. The remainder of the proof parallels the proof of proposition 1.

**Proofs of Propositions 3-7.** See the companion technical appendix.

**Proof of Proposition 8.** Denote by $t^*$ the (least, if there exist many) solution of the equation $B(t) = L(t)$ or $t^* = t$ if $L \leq B(T)$ or $t^* = 0$ if $L \geq B(0)$. When the boundary curve is only left-continuous and the cap $L$ falls between the two limit points, say $L \in [B(i); s > t], B(t)]$ for some point $t$ of discontinuity, let $t^*$ denote this unique point $t$.

By the same proof as in Proposition 1 with the function $u(S) = (S - L - K)^+$ we know that the exercise region for the constant-cap call option has the form $S_i \geq B^*(t)$ for some boundary $B^*(t)$ and that the function $t \rightarrow B^*(t)$ is nonincreasing. We prove that $B^*(t) \leq B(t)$ for $t \in [0, t^*)$. In fact, if $S_i \geq L$, immediate exercise is clearly optimal since the payoff $L - K$ is maximal. If $L \leq S_i \geq B_i$, using $C^*(S_i, t) \leq C(S, t) = g(S) = g_i^*(S_i)$, where $g_i(S) = (S - K)^+$, $g^*(S) = (S - L - K)^+$ we have $C^*(S_i, t) = g_i^*(S_i)$. Then $B^*(t) \leq B(t)$.

Suppose now that $0 < t^* < T$ and assume that $t \in [t^*, T)$. Define the gap $\Lambda(t) = B(t) - L - B^*(t)$ and note that $\Lambda(t) \geq 0$. If $S_i < B(t)$ the policy of exercising at the first hitting time of the curve $B$ is feasible since $B(u) \leq L$ for any $u \in [t, T)$. Therefore $C^*(S_i, t) \geq C(S, t) > g(S) = g_i^*(S_i)$. Combining this inequality with the reverse inequality enables us to conclude $B(t) + L = B^*(t)$ for $t \in [t^*, T)$, i.e. $\Lambda(t) = 0$. For $t \in [0, t^*]$, since $\Lambda(t) = L - B^*(t)$ is non-decreasing and $\Lambda(t) < 0$, we have $\Lambda(t) = 0$ (since $\Lambda(t^*) = 0$ already). Therefore $B(t) + L = B^*(t)$ as desired.

The proofs for the cases $t^* = 0$ and $t^* = T$ are similar.

**Proof of Proposition 9.** Write $t_i = \arg \max_{s \in [S, T]} t^*\{[L(s) - K]$. By assumption, the function $\Psi(t)$ is increasing over $[0, t_j)$ and decreasing over $[t_j, T)$. We first determine the exercise policy when $t \in (t_j, T)$. In fact, the same proof as in Proposition 8 shows that the exercise region is $S_i \geq L(t) \times B(t)$ for $t \in [t_j, T)$.

Next consider the situation $0 \leq t \leq t_j$. Let $V = \{S \leq L(t), t \in [0, T]\}$. Without loss of generality we may assume that $0 < t^* < T$. If $S_i > L(t)$, the policy of exercising at the first hitting time of $V$ dominates the immediate exercise policy since the expected payoff function is increasing on $[0, t_j]$. This means that $S_i$ must be bounded above by $L(t)$ if $S_i$ belongs to the $t$-section of the exercise set at $t \in [0, t_j)$.

**Case 1.** $t \leq t_j$.

(i) Suppose $t \in [t^*, t_j)$, if $S_i \in [B(t), L(t)]$ it is clear that immediate exercise must be optimal. If $S_i < B(t)$, the policy of exercising at the first hitting time of the curve $B$ is feasible for the capped-call option and, as a result, immediate exercise is suboptimal. In other words, the exercise region is $B(t) \leq L(t)$ for $t \in [t^*, T]$.

(ii) Suppose now that $t \in [0, t^*)$. Above we have shown that the exercise region must be contained in the region $[S_i \leq L(t)]$. Suppose $C^*(S, t) = g_i^*(S, t) = (S - L(t) - K)^+$ for some $t \in [0, t^*)$, $K \leq L(t)$. Then, by an argument similar to that in proposition 1, we have that for any $S \leq S_i \leq L(t)$, $C^*(S, t) = g_i^*(S, t)$. This establishes the connectedness of the exercise region for $t \in [0, t^*)$. Moreover, by the same argument as in proposition 1, we may define $B^*(t) = g_i^*(S, t)$ starting from some unknown $t_e \in [0, t^*)$. By construction the exercise region is then, $B^*(t) \leq S \leq L(t)$ when $t_e \leq t \leq t^*$ and $B^*(t)$ is nonincreasing. Define $\Lambda(t) = L(t) - B^*(t)^{+}$. Then $\Lambda(t)$ is increasing since $L(t)$ is increasing and $B^*(t)$ is nonincreasing. Moreover, $\Lambda(t^*) = 0$ by the first paragraph of this proof. Therefore $\Lambda(t) = 0$. Hence the exercise region must be contained in the curve $S_i \leq L(t)$. We conclude that the optimal exercise policy is a $(t_e, t^*, t)$-policy, where $t_e = \inf \{t \in [0, t^*) : C^*(L(t), t) = L(t) - K\}$. This is determined by solving a univariate optimization problem as in Broadie and Detemple (1995).

**Case 2.** $t_j < t^*$. The proof is the same as when $t \in [0, t^*)$ in Case 1. This completes the proof of the proposition.

**Proof of Proposition 10.** The proof is similar to Broadie and Detemple (1996), Theorem 1. It relies on Lemma 1 in their paper, which holds under our more general assumptions.

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