EXECUTIVE STOCK OPTIONS: A FIRM VALUE APPROACH

PHELIM BOYLE
Center for Advanced Studies in Finance, University of Waterloo
Waterloo, Ontario N2L 2G1, Canada
pboyle@uwaterloo.ca

WEIDONG TIAN
Department of Statistics and Actuarial Science, University of Waterloo
Waterloo, Ontario N2L 2G1, Canada
wdtian@uwaterloo.ca

Executive stock options are an important component of executive compensation and the topic is of interest to both practitioners and academics. The vigorous debate on whether these options should be treated as an expense is subsiding but discussion continues on how these instruments should be valued in order to expense them. In this paper, executive stock options are viewed as contingent claims on a firm’s assets and we formalize this through the concept of an augmented balance sheet. This means that the total market value of the firm’s assets is equal to the market value of its traded securities plus the market value of its stock options. This approach leads to two valuation formulae for these options: one in terms of the firm’s stock price and the other in terms of firm value. We explore the connections between these two approaches and derive explicit valuation formulae under certain assumptions.

Keywords: Valuation; executive stock options; dilution.

1. Introduction and Background

This paper discusses the valuation of executive stock options. In particular, we use a firm value approach, which treats these options as liabilities of the firm. The valuation of these instruments has been the subject of intense debate and our approach provides a conceptual valuation framework. We provide a consistent methodology for harmonizing the balance sheet with the income statement. The basic idea behind this approach was first proposed by Sidenius (1996) and extended by Darsinos and Satchell (2002).

To provide some background, we first give a brief description of executive stock options and mention some issues surrounding them. Executive stock options are widely used to compensate senior executives. Core and Guay (2001) estimated that by the late 1990s executive stock options of large US firms represented about 7% of total outstanding shares. These are basically call options granted to executives as part of their remuneration. Normally, there is an initial period called the vesting period during which the employee cannot exercise her options and is required to forfeit them should she leave the company. After the vesting period has elapsed the executive can exercise her options but she is restrained from selling them.

It is instructive to compare executive stock options with warrants, since in some respects they resemble warrants and in other respects they are different. When a firm issues warrants it receives cash which becomes available for investment. When a firm grants stock options it does not receive any cash. Issuing executive stock options reduces the firm’s total
compensation expense and we return to this point later. Whereas warrants are generally exchanged traded securities, stock options are not. When warrants are issued they are recognized explicitly as liabilities on the balance sheet. On the other hand, when executive stock options are granted they are not recognized explicitly on the balance sheet. It could be argued that executive stock options are liabilities of the firm and that they should be recognized as such. When warrants are exercised the firm receives the strike price and issues new shares to the warrant holders. Similarly, when stock options are exercised the firm receives the strike price and issues new shares to the option holders. In both cases the issue of new shares increases the number of outstanding shares. This effect is known as "dilution" and results in a reduction in the proportional stake of outside shareholders in the firm.1

The relationship between the granting of stock options and future firm performance is subject to considerable controversy. One school of thought argues that executive stock options align the interests of the executives with those of the outside shareholders and that they thus provide a powerful incentive for the executives to work hard and increase future earnings and hence firm value. This viewpoint is reflected in several papers including Demsetz and Lehn (1985), Himmelberg et al. (1999), Core and Guay (1999), Rajgopal and Shevlin (2002). In a recent paper Hanlon et al. (2003) find empirical support for the incentive alignment hypothesis.

An alternative perspective is that they increase the potential for moral hazard instead of reducing it. It is contended that senior executives use options to compensate themselves far beyond the level shareholders would view as optimal. Stock options provide a convenient way for managers to extract excessive rents from the firm since their issuance does not involve an immediate cash transfer. Bebchuk et al. (2003) suggest that stock options permit inefficient wealth transfers from shareholders to managers. They argue that rent seeking executives increase their compensation using options rather than cash because it is easier to do so. Johnson et al. (2003) examine the relation between executive compensation and corporate fraud. Furthermore, the recent accounting scandals at Enron, World Com, and Global Grossing have been attributed to the excessive risk taking perhaps partly induced by the perverse features of executive stock options (see Hall and Murphy, 2003).

The incentives provided by stock options can lead to dysfunctional behavior by managers. There is a conflict of interest since managers have inside knowledge about the firm’s future prospects that is not available to outside investors. Yermack (1997) analyzed the stock option awards to CEO’s in cases involving managerial discretion just prior to the release of favorable news about the company and found evidence of manipulation. Companies have used hedging programs to help defray the costs associated with their employee stock options and this provides another opportunity for questionable practices. For example, until January 2000 Microsoft had a program of selling puts on its own stock to help fund its stock option plan. The put premiums collected were used by the company to help defray the cost of buying back Microsoft shares on the open market to meet the needs of its stock option plan. Such a strategy is profitable as long as the stock price is rising. However, if the price of Microsoft falls the puts become in the money and Microsoft becomes liable for the payout on the puts. Interestingly, Microsoft decided to discontinue its stock option plan in January 2000 and the stock price dropped by about 50% during the next 15 months.2

In contrast, the exercise of a standard option does not have any impact on the underlying because a standard option contract is between two independent parties and the stock price is merely used as a reference to determine the option’s value.

2See Microsoft, various annual reports.
There has also been intense debate about the accounting and expensing of executive stock options. Until recently these options were valued at their intrinsic value for accounting purposes. This meant that since most options were granted with a strike price equal to the current stock price their intrinsic value was zero. Hence, they were not recognized as an expense in a company’s financial statements. However, the International Accounting Standards Board through its IRFS 2 has set guidelines for the expensing of executive stock options and national accounting organizations are issuing similar guidelines.

In the US the detailed recommendations for expensing executive stock options are provided in FAS123(R). In March 2005 the SEC provided practical details on how it will interpret the standards in FAS123(R) and gave specific guidance on various aspects of the valuation model. While the controversy still endures, the focus of the debate has shifted from whether these options should be expensed to how this should be done.

There is an extensive academic literature on the valuation of executive stock options. Some authors concentrate on particular institutional features, such as delayed vesting, nontransferability, and absence of a liquid market. See Huddart (1994), Marcus and Kulatilaka (1994), Rubinstein (1995), and Hull and White (2004). Others focus on exercise behavior and incorporate individual risk preferences. It is known that employees tend to exercise grants soon after vesting if the option is sufficiently in the money. See Carpenter (1998), Detemple and Sundaresan (1999), Carr and Linetsky (2000).

A number of recent studies have developed models which assume that the executive values the option using a utility based approach to obtain an estimate of the executive’s own subjective value for the options. These models generate predicted patterns for early exercise behavior which in turn can be used to estimate the cost to the company of granting the option. For example, Chance and Yang (2004) have used an expected utility framework in a binomial setting to study the effects of various inputs on the executive’s valuation of these contracts and hence compute the cost to the corporation. Cai and Vijh (2005) investigate the impact of allowing the executive to hold the market portfolio on the executive’s subjective valuation of the stock options. Bettis (2005) calibrate actual exercise data using a large database and infer plausible parameters for the valuation exercise. This work represents a promising line of attack on the valuation problem since it captures the lack of liquidity and exercise behavior in the valuation model.

The dilution associated with executive stock options has also been examined. Darsinos and Satchell (2002) and Bodurtha (2002) take the relationship between firm value and the stock options as given. The pricing formula for the executive stock option based on firm value is compared with the Black–Scholes formula based on the stock price. Garvey and Milbourn (2001) analyze the impact on the stock return (distribution) when stock options are issued. In these papers, the relationship between the market value of the firm and the value of the executive stock options is taken as given.

The purpose of the current paper is to develop a conceptual framework for the valuation of stock options that abstracts from many of the actual institutional details. These features although of great importance in practice are not central to the main point we wish to make in this paper. Our objective is to provide a conceptual framework for the valuation of executive stock options and discuss the valuation in terms of the stock price as well as in terms of firm value. As noted above there are several papers in the literature which do take account of the different institutional characteristics of executive stock options and explain how these features affect their valuation.

The layout of the paper is as follows. Section 2 describes the augmented balance sheet approach. The augmented balance sheet equation always holds irrespective of how firm value is affected by the issuance of executive stock options. Section 3 develops a theoretical framework based on the firm value approach. We
present two different pricing functions, one based on the stock price the other based on firm value. We also discuss calibration procedures. Section 4 examines the relationship between the two pricing functions. We discuss the valuation of executive stock options in our framework and compare the results of the two pricing functions. Section 5 concludes. Proofs are provided in Appendix A.

2. Augmented Balance Sheet

It is useful to start with a very simple example. We assume an all equity firm. We first suppose this firm issues a single tranche of warrants. The cash received from selling these warrants is used for investment purposes. Initially, assume that the firm value increases by the amount of cash it receives. On the liability side the firm now has both common stock and warrants. We should stress that we follow the convention here of referring to all claims as liabilities. The market value of the firm’s liabilities is equal to the sum of the market value of the firm’s common stock plus the market value of the firm’s warrants. After the issue the market value of the firm’s assets remains equal to the total market value of the firm’s liabilities.

Note that if the proceeds from the warrants were used to make a positive net present value project, this would increase the total market value of the firm by an amount greater than the cash received for the warrants. In this case, the firm value would be reflected in the market prices of the firm’s traded securities—stocks and warrants. The market value of the firm’s assets in this case would still be equal to the total of the firm’s liabilities. The same conclusion holds if the proceeds from the warrants are used by management for a negative net present value project. This would reduce firm value but it will still be the case that the market value of the firm’s assets is equal to the market values of its liabilities. In each case, the forces of no arbitrage provide the market discipline that equates the market value of assets to the market value of the liabilities.

This example illustrates how we approach executive stock options.

These options are issued to remunerate executives and at the same time provide them with incentives to maximize the share price. In contrast to warrants when such options are issued there is no immediate cash flow to the firm. Once they are issued they become a liability of the firm. Bodie, Kaplan and Merton (2003) also emphasize this point. Just like warrants, executive stock options are contingent claims on future firm value. However, unlike warrants, they are not recorded on the balance sheet as liabilities of the firm. But this accounting convention does not change the underlying reality that they represent contingent claims on the future value of the firm.

This plays an important role in our subsequent analysis. Suppose an all equity firm has issued a number of executive stock options. After they are issued will the total market value of the firm be equal to the total market value of the firm’s common stock? We suggest that in a perfect frictionless market the answer to this question is no. The reason is that the firm has in addition to its common shares the contingent claims represented by the firm’s options. In this respect, these contingent claims are like warrants. The accounting convention for warrants is to put the liabilities on the balance sheet. Traditionally, the liabilities associated with executive stock options have only been recognized through footnote disclosures.

In our framework, once the options have been granted, the market value of the firm is equal to the market value of common stock plus the market value of the options just as in the warrant case. We call the revised balance sheet, which now includes the market value of the options the augmented balance sheet. Note that the total of the assets is still equal to the equal liabilities for the augmented balance sheet. However, does this approach provide an opportunity for arbitrage profits? We suggest not. In order to capture arbitrage profits an investor would have to sell the asset and extinguish all the executive stock options. We cannot see this happening. These

---

4This accords with UK accounting conventions. We thank the referee for highlighting this point.
options are contractual obligations of the firm and cannot be so lightly extinguished.

We can illustrate this point using a simple example, which is based on the one given by Bodie et al. (2003). Suppose we have two companies denoted by M and K. They have the same assets and are identical in every way except in how they compensate their executives. Both are all-equity firms, and they have no other liabilities.

Company K pays its executives $1,000,000 in total compensation in the form of cash at the end of the year. At the end of the year it also issues warrants. The market value of the warrants is $400,000. These warrants cannot be exercised for 1 year. Let us assume that Company K also requires that each executive uses 40% of their salary compensation to buy these warrants. So, the total amount the executives will spend on the warrants is $400,000. The executives now own all the warrants. After the warrants are issued Company K has a new liability on its balance sheet corresponding to these warrants. Company K’s net cash outflow is $600,000. The executives have a package of $600,000 in cash and $400,000 in warrants.

Company M, on the other hand, pays its executives a package consisting of $600,000 cash and $400,000 worth of executive stock options. As Bodie et al. (2003) note, the economic position of the two companies is identical even if this is not reflected in conventional accounting practice. Each company has paid $600,000 in cash compensation plus options worth $400,000. Suppose that the total assets of each company before any compensation expense is $21 million. After Company K pays its executives, the market value of K’s assets is $20.4 million. On the liability side its warrants are worth $400,000 and so its common stock is now worth $20 million.

The value of M’s assets is $21 million before it pays its executives. After it pays them, the firm’s asset value falls to $20.4 million. M has also taken on the liability to pay its stock options. The market value of M’s stock should also be equal to 20 million since both firms are identical in economic terms. If we augment Company M’s balance sheet to reflect the $400,000 worth of new executive stock options, then economic realism is restored. Company M has assets valued at $20.4 million, common share worth $20 million, and stock options worth $400,000.

The augmented balance sheet argument can also be explained as follows. Suppose an all-equity firm has issued executive stock options to its executives. To cover this liability, it buys, from an investment bank, an option contract that has the same features as the options issued to its executives. Once the firm pays for the contract its assets will fall by the amount paid to the investment bank. After completing this transaction, the firm has discharged its liability under the compensation contract and the firm’s only liability is its common stock. The value of this stock must be equal to the value of the firm’s assets, which in turn is equal to the original firm value minus the cost of the options. This means that before the transaction with the investment bank the firm’s assets are equal to the value of the stock plus the value of the options.

Our augmented balance sheet relationship holds irrespective of whether the granting of stock options increases or reduces firm value. There is no consensus in the literature on this point. Based on a simple view of the world we can identify three cases.

Case One: Firm Value increases when executive stock options are issued. This corresponds to the incentive alignment hypothesis which we discussed earlier.

Case Two: Firm Value decreases when executive stock options are issued. This corresponds to managerial rent extraction hypothesis. In this case, the stock options provide a vehicle for managers to extract rents from outside shareholders.

Case Three: Firm Value is unchanged when executive stock options are issued. Here, we assume

Note that this approach glosses over the important issue of endogeneity. Suppose, for example, we truly did have optimal contracts and that option compensation was selected optimally along with everything else to maximize shareholder wealth. Then if we were at a maximum and perturbed the number of options granted either upwards or downwards this could decrease firm value in either case.
that the granting of stock options does not affect firm value.

Note that in all three cases after the options have been granted the augmented balance sheet relationship holds. We assume that the impact of issuing the options is reflected in the market value of the firm. In each case, the revised market value of the firm is equal to the market value of the firm’s stock plus the value of the firm’s executive stock options.

3. Firm Value Approach

In this section we develop the firm value approach more formally. We make the following assumptions. These assumptions enable us to present the model in a simple setting.

- **Assumption One:** The firm in question is an all equity firm with no outstanding debt.
- **Assumption Two:** Perfect Capital Markets — there are no transaction costs, taxes, short selling restriction, or other such frictions. Trading takes place continuously in time. The borrowing and lending interest rates are equal.
- **Assumption Three:** The term structure of interest rates is flat and known with certainty.
- **Assumption Four:** The firm just issues one type of executive stock option. These options all mature at the same time and have the same strike price. Each option confers the right to buy a single share of underlying stock.
- **Assumption Five:** All the options are exercised at the same time if exercise occurs.

Assumption One could be relaxed to include debt in the capital structure. Assumption Two is fairly standard. Assumption Three could be extended to deterministic interest rates. Assumption Four could also be relaxed to cover the case where there are different executive stock options with different maturities and strikes. In practice, an executive stock option cannot be exercised until after the vesting period. We could, for example, assume that the maturity of the option is the expected life of the option. We note that tax considerations are also important in the context of stock options but we do not model them in this paper. It is possible to take them into account in the firm value approach.\(^7\) Assumption Five is strong because we assume that the option premium is the same for any individual employee.

We use the following notation:

- \(S:\) the current price of the firm’s stock
- \(n_s:\) the number of outstanding shares of stock
- \(n_e:\) the number of executive stock options outstanding
- \(T:\) the time to option maturity
- \(K:\) the strike of the options
- \(e:\) the current market value of each option
- \(X:\) denotes the total current value of the options; \(X = n_e e\)
- \(V:\) the total value of the firm
- \(v:\) denotes the firm value per common share, that is \(v = V / n_s\)
- \(\lambda:\) the dilution factor,\(^8\) defined as

\[
\lambda = \frac{n_e}{n_s + n_e}
\]

The firm value is equal to the total market value of the firm’s assets. We have

\[
V = n_s S + n_e e \quad (1)
\]

Thus, it follows that

\[
v = S + \frac{n_e e}{n_s} \quad (2)
\]

Now, consider the payoff on a single executive stock option. At any time \(T_1 \leq T\), the payoff is \((S_{T_1} - K)\), if the option is exercised,\(^9\) otherwise it is zero. Hence, the payoff in terms of underlying stock is \((S_{T_1} - K)\). From the firm value perspective, if the options are exercised, the firm value increases by \(n_e K\), while the number of shares increases by \(n_e\). The stock price after exercise becomes

\[
S_{T_1} = \frac{V_{T_1} + n_e K}{n_s + n_e}
\]

\(^6\)This is consistent with the FASB’s proposal 148 but it is only an approximation. This feature could be modelled more precisely.

\(^7\)See, for example, Leland (1994).

\(^8\)There are different ways of defining the dilution factor. We follow the convention of Sidenius (1996).

\(^9\)We are assuming that exercise will only take place if the option is in the money.
Hence, the payoff on the option at time $T_1$ equals

$$S_{T_1} - K = \lambda(v_{T_1} - K)$$

So, in terms of firm value per share, the payoff equals $\lambda((v_{T_1} - K)^+)$. Since $T_1$ can be any time $\leq T$, the payoff can be written as either $(S - K)^+$ or $\lambda(v - K)^+$. We denote the option price function in terms of the underlying stock $S$, as

$$e = D(S)$$

We can also find a formula for the option price in terms of the firm value. The steps involved are as follows:

$$e = D(S) = e^{-rT}E^Q[(S_{T_1} - K)^+]$$

$$= e^{-rT}E^Q[\lambda(v_{T_1} - K)]^+$$

$$= \lambda e^{-rT}E^Q[(v_{T_1} - K)]^+$$

$$= \lambda C(v)$$

Hence, we have the following expression for the option price in terms of the firm value per share:

$$e = \lambda C(v)$$

From the employee’s point of view, the optimal exercise policy depends on the individual’s risk preferences and other considerations. However, under our assumptions, which consider these options as a block, the executive stock options can be treated as American-type options after the expiration of the vesting period. Therefore,

$$e \equiv D(S) = \text{Sup}_\tau E^Q[(S_\tau - K)^+]$$

and

$$e \equiv \lambda C(v) = \text{Sup}_\tau E^Q[\lambda(v_\tau - K)^+]$$

where $\tau$ is any stopping time in $(0, T]$ and $Q$ denotes the risk-neutral measure.\(^{10}\)

In practice, executive stock options are usually treated as European-style\(^{11}\) options and, in this case, we have

$$D(S) = e^{-rT}E^Q[(S_T - K)^+]$$

$$C(v) = e^{-rT}E^Q[(v_T - K)^+]$$

However, the following relationship between the two different pricing functions holds for both European and American specifications.

**Proposition 1.** The relationship between the two pricing functions is given by the pair of functional equations

$$\lambda C(v) = D(v - (1 - \lambda)C(v))$$

$$D(S) = \lambda C(S + \left(\frac{1}{\lambda} - 1\right)D(S))$$

Proof. See Appendix A, or Darsinos and Satchell (2002). □

This proposition states that, given the pricing function $C(v)$ in terms of firm value per share, $v$, we can obtain the pricing function $D(S)$ in terms of the stock price $S$. Or given the function $D(S)$, it is possible to obtain the function $C(v)$. To put it another way, given the stock price distribution, it is possible to derive the distribution of the firm value. Conversely, given the distribution of the stock price, the distribution of the firm value can be obtained. The distribution of the firm value and stock price determine each other. This proposition implies that it is not possible to have a lognormal distribution for both $v$ and $S$.\(^{12}\)

Some additional comments on the relationship between Eqs. (8) and (9) may be helpful. It is not easy to solve both functions $C(\cdot)$ and $D(\cdot)$ from these two equations. Given the function $C(\cdot)$, Eq. (9) is an implicit equation for the function $D(\cdot)$ since $D(\cdot)$ appears on both sides of this equation. Solving $D(\cdot)$ amounts to the identification of a fixed point where the value of the option must be the same from both perspectives.

Sidenius (1996) and Darsinos and Satchell (2002) show that the above proposition also holds for warrants. However, when a firm issues warrants there is an immediate infusion of cash

\(^{10}\)Even though the firm value cannot be traded directly, it can be replicated using the underlying stock. Therefore, we can use the firm value as a state variable in the same way as a tradable underlying share. This approach has its heritage in Merton’s paper (1974) on the valuation of risk debt. An alternative justification is provided by Erickson and Remy (2004) who show this approach can be applied as long as one of the firm’s securities is tradable.

\(^{11}\)See FASB No 123, No 148, Rubinstein (1995), Hull and White (2004), etc.

\(^{12}\)This result is well-known when the firm issues warrants. See Galai and Schaefer (1978).
which increases firm value. There is no impact on the underlying stock.\footnote{See Galai and Schneller (1978), Emanuel (1983).}

\subsection{Calibration}

To price options we can make assumptions about the distribution of firm value or the stock price distribution. It is common to make assumptions about the stock price. Often it is assumed that the stock price follows a lognormal distribution. When the dilution factor $A$ is close to 1, i.e. not many options are granted compared with number of outstanding shares, the impact can be ignored. Hull and White (2004) also argue that we may still use a lognormal assumption on the stock price, even when many options are issued, if we choose the market value of the stock after issue. It is presumed that this stock price reflects the impact of issuing the options.

On the other hand, if there are a large numbers of options, the outside shareholders are saddled with a heavy future liability. It is sometimes argued that when the stock reaches some level, options will be exercised and this could in turn influence the stock price distribution. Such a return pattern suggests that the Black–Scholes lognormal assumption might not be a good assumption for pricing these options.

If we are willing to assume that, as a first approximation, firm value does not change when the options are issued, we can make assumptions about the stochastic process for firm value and use it as the state variable. However, the firm value is not a traded security. This makes calibration harder than if we used the stock price. We now explain how to calibrate the firm value model using observable market information from the stock price process.

Assume the firm value per share $v$ follows the following log-normal process.\footnote{This assumption is reasonable because there is only one type of option granted by our Assumption Four, so no “smile” feature is required. We could take account of the “smile” with different maturities and different strikes by using other processes for $v$. This extension is not addressed here.}

\begin{equation}
\frac{dv}{v} = \left[ \mu - \delta \right] dt + \sigma v dZ \tag{10}
\end{equation}

where $\mu$ is the instantaneous expected rate of return on the firm per unit time, $\delta$ corresponds to the dividend rate, $\sigma$ is the volatility of the process, and $dZ$ denotes a standard Brownian motion. As Merton (1974) has shown, $\mu$ could be replaced by the instantaneous short rate to price contingent claims on the firm value. Ericsson and Reneby (2004) show that, we can always write the firm value in terms of the risk-neutral measure because the stock is tradable. Therefore, we can represent the firm value process as

\begin{equation}
\frac{dv}{v} = \left[ r - \delta \right] dt + \sigma v dZ^Q \tag{11}
\end{equation}

where $Z^Q$ denotes the Brownian motion under the equivalent martingale measure corresponding to a risk-neutral world. We now explain how to estimate $\sigma$.

Let $\sigma$ be the implied volatility of the underlying stock. Recall the relationship:

\begin{equation}
S = v - (1 - \lambda)c(v) \tag{12}
\end{equation}

Using Ito’s lemma we obtain

\begin{equation}
S\sigma = \left[ 1 - (1 - \lambda)c'(v) \right] v \sigma \tag{13}
\end{equation}

where $C'(v)$ denotes the first derivative of the option with respect to $v$. Thus, using (12) again, we obtain

\begin{equation}
\sigma = \frac{v - (1 - \lambda)c(v)}{v(1 - (1 - \lambda)c'(v))} \tag{14}
\end{equation}

where $\sigma$ also appears on the right-hand side (in the terms $C(v), C'(v)$) of the above formula (14). Therefore, a numerical method is required to solve for $\sigma$. Note that this calibration procedure is similar to the calibration of the Merton model (Merton, 1974) corporate debt using stock market information. In the Merton model, the stock is viewed as the contingent claim on the underlying firm value. Thus, the stock’s implied volatility can be used to derive the implied volatility of the firm.
4. Discussion

To price executive stock options we need to make assumptions about the distribution of certain state variables. In our framework, the state variable could be either the stock price $S$ or the firm value per share $v$. Given the distribution of $v$, the pricing function $C(v)$ is available. In this case the pricing function $S$ is determined by the functional equation as in Proposition 1. By the same reasoning if we know the distribution of the stock price, the pricing function $D(S)$ is available as well. Then, it is possible to derive the pricing function $C(v)$ in terms of the firm value per share. However, the functional equations (8) and (9) are in general difficult to solve. For the rest of this section the discussion is based on the functional equations in Proposition 1.

4.1. Pricing functions for European-type options

We assume the option is European. This makes it possible to derive closed-form pricing functions. Our next result explains how to derive the pricing functions from one another using the martingale approach.

Proposition 2

(1) Assuming the distribution of $v$ is given in the risk-neutral world, then,

$$D(S) = e^{-rT} E^Q[(f(vT) - K)^+]_{S=v}$$(15)

where $Q$ denotes the risk-neutral measure, $f(x) = x - (1 - \lambda)C(x)$, $\Phi$ is the inverse of $f(\cdot)$.

(2) Assuming the distribution of $S$ is given in the risk-neutral world, then,

$$C(v) = e^{-rT} E^Q[(g(S_T) - K)^+]_{S=g(v)} $(16)

where $Q$ denotes the risk-neutral measure, $g(x) = x + (\frac{1}{\lambda} - 1) D(x)$, and $\phi$ is the inverse of $g(\cdot)$.

Proof. See Appendix A.

This proposition is useful for obtaining the option pricing functions for any distribution. Analytical pricing formulae are available for special classes of distributions as we will see in the next proposition. For multi-dimensional distributions, Monte Carlo simulation could be used but we do not pursue this in the current paper. Here is a simple example to illustrate how this proposition can be used for general distributions.

Assuming $v$ has the following process (in the risk-neutral world):

$$\frac{dv}{v} = r\, dt + \sigma(v, t)dZ$$

Then $C(v)$ is known. We wish to calculate $D(S)$ in terms of the current stock price $S$. Here are the steps involved.

Step 1. Beginning with the current stock price $S$, calculate $v = \Phi(S)$.

Step 2. Starting from the current value $v$, generate $v_T$ by using Eq. (17) via simulation.

Step 3. For each scenario use $v_T$ to calculate $f(v_T)$, and hence $(f(v_T) - K)^+$.

Step 4. Average the results in Step 3, and multiply by $\exp(-rT)$ to obtain $D(S)$.

In the special case when $v$ has a lognormal distribution, we can derive an analytical formula for $D(S)$.

Proposition 3. Assuming $v$ is lognormal, then,

$$D(S) = vN\left(\frac{\ln(v/\Phi(K)) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

$$-Ke^{-rT}N\left(\frac{\ln(v/\Phi(K)) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

$$-\left(1 - \lambda\right)A\left(\frac{\ln(\Phi(K)/v) - \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

$$\frac{\ln(v/\Phi(K)) + 2rT - \sigma^2T}{\sigma\sqrt{T}}$$

$$+(1 - \lambda)Ke^{-2rT}A\left(\frac{\ln(\Phi(K)/v) - \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

$$-\frac{\ln(v/\Phi(K)) + 2rT - \sigma^2T}{\sigma\sqrt{T}}$$
where \( v = \Phi(S) \), \( N(\cdot) \) is the cumulative normal distribution function, \( n(\cdot) \) is the normal density function, and
\[
A(a, b) = \int_a^\infty n(x)N(x + b)\,dx
\]
for all \( a, b \).

**Proof.** The proof is available from the authors. It is also possible to derive an analytical formula for \( C(v) \) if we assume that \( S \) is lognormal.

### 4.2. Comparisons of the pricing functions

In this section, we will compare the pricing functions \( C(\cdot) \) and \( D(\cdot) \).

**Proposition 4.** For any dilution factor \( \lambda \), we have
\[
\lambda C(S) \leq D(S) \leq C(S)
\]
In particular, when \( \lambda \) gets close enough to 1, then
\[
C(S) \sim D(S)
\]
**Proof.** See Appendix A.

The first part of this proposition is easy to see. Recall that the option premium is increasing with respect to the underlying, and the option is non-negative. Hence,
\[
\lambda C(S) \leq \lambda C(v) = D(S)
\]
The second inequality, however, involves a more subtle property of options.

By using this proposition, if \( \lambda \) is close to 1, both functions \( C(\cdot) \) and \( D(\cdot) \) are close.\( 16 \)

The relationship between the two functions \( C(\cdot) \) and \( D(\cdot) \) is useful for addressing some practical issues. One usually makes assumptions about the stock price distribution. In this case, the option premium becomes \( C(S) \). As we noted earlier, there might be occasions when it would be reasonable to make the distributional assumption about the firm value. In this case, the option premium becomes \( v = \lambda C(v) \), which is \( D(S) \).

\( 16 \) This is consistent with Bodurtha’s (2002) empirical work.
pricing functions in terms of the firm value and in terms of the stock price and we discussed the connection between the firm value distribution and the stock price distribution. To derive our results we made simple assumptions about the firm’s capital structure. We also ignored several important institutional features. In a more realistic model, capital structure could be extremely complicated. The firm’s liabilities could include long-term, medium-term and short debt, warrants, convertible bonds, and restricted stock. Many types of executive stock options and warrants are available across different strikes and maturities, and in the real world options are sometimes repriced. Nevertheless, despite these limitations it is hoped our approach provides a useful perspective on executive stock options.

Appendix

Proof of Proposition (1)

Combining Eqs. (2) and (4) we have

\[ v = S + \frac{\lambda}{\lambda} C(v) = S + (1 - \lambda) C(v) \quad (A.1) \]

Then

\[ S = v - (1 - \lambda) C(v) \quad (A.2) \]

Now, using formula (A.1) and (3), we have

\[ v = S + \left( \frac{1}{\lambda} - 1 \right) D(S) \quad (A.3) \]

Therefore,

\[ \lambda C(v) = e = D(S) = D(v - (1 - \lambda) C(v)) \]

and

\[ D(S) = e = \lambda C(v) = \lambda C \left( S + \left( \frac{1}{\lambda} - 1 \right) D(S) \right) \]

Proof of Proposition (2)

Denote by \( f_{R\mathcal{N}}(v; \mathcal{Q}) \) the probability density function of \( v \) given \( \mathcal{Q} \). Then,

\[ f_{R\mathcal{N}}(S_t; \mathcal{Q}) = \frac{f_{R\mathcal{N}}(v \equiv \Phi(S_t); S)}{\partial S/\partial v} \quad (A.4) \]

Then

\[ D(S) = \exp(-rT) \int_{K}^{S} (S_r - K) f_{R\mathcal{N}}(S_r; S) \text dS_r \]

\[ = \exp(-rT) \int_{K}^{S} (S_r - K) f_{R\mathcal{N}}(\Phi(S_r); S) \text dS_r \]

\[ = \exp(-rT) \int_{K}^{S} \left( f(v) - K \right) \]

\[ \times \int_{\Phi(K)}^{\Phi(S_r)} \left( \frac{\partial S_r}{\partial v} \right) \text d\zeta \]

\[ = \exp(-rT) \int_{K}^{S} \left( f(v) - K \right) f_{R\mathcal{N}}(v; \mathcal{Q}) \text d\zeta \]

\[ = \exp(-rT) e^{rT} (f(v) - K) \mid_{v=\Phi(S)} \]

where the first line comes from the definition of \( D(S) \), the second one from (A.4), the third from the transformation \( S = f(v) \). Hence (1) is proved. The proof of (2) is similar. It suffices to show that the function \( g(\cdot) \) has an inverse. By Eqs. (A.1) and (A.3), \( g \) is essentially the function \( \Phi \). So \( f \) is the inverse function of \( g \).

Proof of Proposition (4)

Using the mean-value theorem on formula (9), we have

\[ D(S) = \lambda \left[ C(S) + C'(\zeta) \left( \frac{1}{\lambda} - 1 \right) D(S) \right] \]

\[ = \lambda C(S) + C'(\zeta)(1 - \lambda) D(S) \]

for some \( \zeta \) between \( S \) and \( S + (\frac{1}{\lambda} - 1) D(S) \). Therefore, we obtain

\[ D(S) = \frac{\lambda}{1 - (1 - \lambda) C'(\zeta)} C(S) \quad (A.5) \]

To prove the second inequality, from (A.3), it is enough to prove that

\[ \frac{\lambda}{1 - (1 - \lambda) C'(\zeta)} \leq 1 \quad (A.6) \]

which is equivalent to

\[ (1 - \lambda) C'(\zeta) \leq 1 - \lambda \quad (A.7) \]

which in turn is equivalent to

\[ C'(\zeta) \leq 1 \quad (A.8) \]

This last equation, (A.8) means that the delta of the option is less than 1. In other words, the option premium is less sensitive than the underlying asset to the price of the underlying. In the
case of the log-normal distribution for the underlying, this is obvious from the Black–Scholes formula for European options. It is also proved by Broadie and Detemple (1995) for American call options. For more general diffusion process, the result can be obtained from Bergman et al. (1996) in the case of European-type options, and from Detemple and Tian (2002) in the case of American-type options.

Acknowledgments
The authors are grateful for support from the Natural Sciences and Engineering Research Council of Canada. We also thank an anonymous referee.

References


